Bayesian Non parametrics - Completely Random Measure (CRM)

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Sampling Normalised Random Measure

$$\mathbb{E}\left[\int f(x)\mu(\mathrm{d}x)\right] \equiv \int \underbrace{\left[\int f(x)\mu(\mathrm{d}x)\right]}_{\mathbb{E}_\mu[f(x)]} \mathrm{d}\mu$$

When this generic $\mu(\mathrm{d}x) \equiv N(\mathrm{d}x)$
Palm Formula:

$$\mathbb{E}\left[\int f(x)G(x,N)N(\mathrm{d}x)\right] = \int \mathbb{E}[G(x,\delta_x + N)]f(x)\lambda(\mathrm{d}x)$$

$$\begin{aligned}
&\mathbb{E}_\mu\left[\exp^{-\int(f(x)+u)\mu(\mathrm{d}x)} \prod_{j=1}^{k} \mu\left(\mathrm{d}x_j^*\right)^{n_j}\right] \\
=&\mathbb{E}_\mu\left[\underbrace{\exp^{-\int(f(x)+u)\mu(\mathrm{d}x)} \mu\,(\mathrm{d}x_k^*)^{n_k}} \prod_{j=1}^{k-1} \mu\left(\mathrm{d}x_j^*\right)^{n_j}\right] \\
=&\mathbb{E}_\mu\left[\left(\int \delta_{x_k^*} \exp^{-\int(f(x)+u)\mu(\mathrm{d}x)\mu\left(\mathrm{d}x_k^*\right)^{n_k}}\right) \prod_{j=1}^{k-1} \mu\left(\mathrm{d}x_j^*\right)^{n_j}\right]
\end{aligned}$$

$$\Pr(\Pi = \pi, \underbrace{\{\theta_c^* \in d\theta_c : c \in \pi\}}_{\theta_1, \theta_2, \ldots, \theta_{|\pi|}} | \mu)$$

$$= \prod_{c \in \pi} \tilde{\mu}(d\theta_c)^{|c|} = \frac{\prod_{c \in \pi} \mu(d\theta_c)^{|c|}}{T^n} = \underbrace{\int_u \frac{1}{\Gamma(n)} u^{n-1} \exp^{-Tu} du}_{T^{-n}} \prod_{c \in \pi} \mu(d\theta_c)^{|c|}$$

This is because:

$$\int_u \frac{1}{\Gamma(n)} u^{n-1} \exp^{-Tu} du = T^{-n} \underbrace{\int_u \frac{T^n}{\Gamma(n)} u^{n-1} \exp^{-Tu} du}_{1} = T^{-n}$$

Therefore, we write:

$$\Pr(\Pi = \pi, \{\theta_c^* \in d\theta_c : c \in \pi, \underline{U \in du}\} | \mu)$$
$$= \int_u \frac{1}{\Gamma(n)} u^{n-1} \exp^{-Tu} du \prod_{c \in \pi} \mu(d\theta_c)^{|c|}$$

Sampling Normalised Random Measure

It can be deduced that,

$$= \frac{1}{\Gamma(n)} u^{n-1} \exp^{-Tu} \mathrm{d}u \prod_{c \in \pi} \mu(\mathrm{d}\theta_c)^{|c|}$$

$$\begin{aligned}
&\Pr(U \in \mathrm{d}u | \mu) \\
&\propto \frac{1}{\Gamma(n)} u^{n-1} \exp^{-Tu} \mathrm{d}u \underbrace{\prod_{c \in \pi} \mu(\mathrm{d}\theta_c)^{|c|}}_{\text{has no } u} \\
&= \frac{1}{\Gamma(n)} u^{n-1} \exp^{-Tu} \mathrm{d}u \\
&= \mathrm{Gamma}(\underbrace{n}_{\text{shape}}, \underbrace{T}_{\text{rate}}) \qquad \text{the second of wikipedia definition}
\end{aligned}$$

Sampling Normalised Random Measure

$$
\begin{aligned}
&\Pr(\Pi = \pi, \{\theta^*_c \in \mathrm{d}\theta_c : c \in \pi\}, U \in \mathrm{d}u | \mu) \\
&= \int_\mu \underline{\Pr(\Pi = \pi, \{\theta^*_c \in \mathrm{d}\theta_c : c \in \pi\}, U \in \mathrm{d}u | \mu)} \Pr(\mu)\mathrm{d}\mu \\
&= \int_\mu \underline{\frac{1}{\Gamma(n)} u^{n-1} \exp^{-Tu} \mathrm{d}u \prod_{c \in \pi} \mu(\mathrm{d}\theta_c)^{|c|}} \Pr(\mu)\mathrm{d}\mu \\
&= \frac{1}{\Gamma(n)} u^{n-1} \int_\mu \underline{\exp^{-Tu} \mathrm{d}u \prod_{c \in \pi} \mu(\mathrm{d}\theta_c)^{|c|}} \Pr(\mu)\mathrm{d}\mu
\end{aligned}
$$

Infinite Divisibility

$$Y = \left(\sum_{j=1}^{n} Y_j^{(n)} \right) \sim p(y) \qquad\qquad Y_j \sim p^{(n)}(y)$$

- As n increases, increments becomes finer, $p^{(n)}(x)$ also change
- But $Y \sim p(x)$ remain unchanged.

For example:

$$Y = \left(\sum_{j=1}^{n} Y_j^{(n)} \right) \sim \text{Pois}(1) \qquad\qquad Y_j^{(n)} \sim \text{Pois}(1/n)$$

Sum of two random variables:

Sum of two (not necessarily independent) random variables: $Z = X + Y$

$$
\begin{aligned}
\Pr(Z \leq z) = \Pr(X + Y \leq z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} p_{X,Y}(x, y) \mathrm{d}x\mathrm{d}y \\
p_Z(z) &= \frac{\mathrm{d}}{\mathrm{d}z} \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} p_{X,Y}(x, y) \mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}z} \int_{-\infty}^{z-y} p_{X,Y}(x, y) \mathrm{d}x \right) \mathrm{d}y \\
&= \int_{-\infty}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}z} \Pr(z - y, y) \right) \mathrm{d}y \\
&= \int_{-\infty}^{\infty} p_{X,Y}(z - y, y) \mathrm{d}y
\end{aligned}
$$

When X and Y are independent:

$$
p_Z(z) = \int_{-\infty}^{\infty} p_X(z - y) f_Y(y) \mathrm{d}y \qquad P(Z = z) = \sum_{k=\infty}^{\infty} P(X = k) P(Y = z - k)
$$

Sum of two independent random variables:

N-Fold convolution $S_n = X_1 + X_2 + \cdots + X_n$, **n-fold convolution**
$f_{S_n(x)} = \left(f_{X_1} * f_{X_2} * \ldots * f_{X_n}\right)(x)$

$$P_y(Y) = \left(p^{(n)}(Y) * p^{(n)}(Y) * \ldots * p^{(n)}(Y)\right)(Y)$$

Moment Generation function:

$$\begin{aligned} M_{x+y}(t) &= \mathbb{E}_{p_{X,Y}(x+y)}[\exp(t(x+y))] \\ &= \mathbb{E}_{p_{X,Y}(x+y)}[\exp(tx)\exp(ty)] \\ &= \mathbb{E}_{p_X(x)}[\exp(tx)]\mathbb{E}_{p_Y(y)}[\exp(ty)] \\ &= M_x(t)M_y(t) \end{aligned}$$

If $M(t)$ is differentiable at zero, then n^{th} moments about the origin are given by $M^{(n)}(0)$:

$$\begin{aligned} M_x(t) &= \mathbb{E}_{p_X(x)}[\exp(tx)] \implies M_x(0) = \mathbb{E}_{p_X(x)}[\exp(0x)] = 1 \\ M_x'(t) &= \mathbb{E}_{p_X(x)}[x\exp(tx)] \implies M_x'(0) = \mathbb{E}_{p_X(x)}[x\exp(0x)] = \mathbb{E}_{p_X(x)}[x] \\ M_x''(t) &= \mathbb{E}_{p_X(x)}[x^2\exp(tx)] \implies M_x''(0) = \mathbb{E}_{p_X(x)}[x^2\exp(0x)] = \mathbb{E}_{p_X(x)}[x^2] \\ &\dots \\ M_x^{(n)}(t) &= \mathbb{E}_{p_X(x)}[x^n\exp(tx)] \implies M_x^{(n)}(0) = \mathbb{E}_{p_X(x)}[x^n\exp(0x)] = \mathbb{E}_{p_X(x)}[x^n] \end{aligned}$$

$$\begin{aligned} \mu &= M_x'(0) \\ \sigma^2 &= \mathbb{E}_{p_X}[(x-\mu)^2] = \mathbb{E}_{p_X}[x^2] - 2\mu\mathbb{E}_{p_X}[x] + \mu^2 = \mathbb{E}_{p_X}[x^2] - (\mathbb{E}_{p_X}[x])^2 \\ &= M_x''(0) - (M_x'(0))^2 \end{aligned}$$

Cumulant Function or Characteristic Equation

Cumulant function $\mathbb{C}(.)$ of $Y = Y_1 + Y_2 + \cdots + Y_t$ $\qquad$ $Y_1 \dots Y_t \sim p(Y_1)$

$$
\begin{aligned}
\mathbb{C}_Y(\theta) &= \ln\left(\phi_y(x)\right) \\
&= \ln\left(\underbrace{\mathbb{E}_{P(Y_t)}[\exp^{i\theta y_t}]}_{\phi_y(\theta)}\right) \qquad \text{where } \phi_y(\theta) \text{ is characteristic equation} \\
&= \ln\left(\mathbb{E}_{P(Y_1,\dots,Y_t)}\left[\exp^{i\theta(y_1+\cdots+y_t)}\right]\right) \\
&= \ln\left(\mathbb{E}_{P(Y_1,\dots,Y_t)}\left[\exp^{i\theta y_1}\dots\exp^{i\theta y_t}\right]\right) \\
&= \ln\left(\left(\mathbb{E}_{P(Y_1)}[\exp^{i\theta y_1}]\right)^t\right) \\
&= t\ln\left(\mathbb{E}_{P(Y)}\left[\exp^{i\theta y_1}\right]\right) \\
&= t\mathbb{C}_{Y_1}(\theta)
\end{aligned}
$$

Levy-Khintchine representation

- Suppose Y is a Levy process with non-negative increments, the kumulant function can be written as:

$$\mathbb{C}_{Y_1}(\theta) = -a\theta + \int_0^\infty \left(\exp^{-\theta y} - 1\right) \nu(\mathrm{d}y)$$
$$\implies \phi_Y(\theta) = \exp\left(-a\theta + \int_0^\infty \left(\exp^{-\theta y} - 1\right) \nu(\mathrm{d}y)\right)$$

- where $a \geq 0$ and ν is a measure on R^+ such that:

$$\int_0^\infty \min\{1, y\}\nu(\mathrm{d}y) < \infty$$

think about the function of $\min\{1, y\}$

- Kumulant function of **all** non-negative Levy processes can be written in this form.
- **important** Non-negative Levy processes are completely determined by a and the Levy measure ν

Compound Poisson Process

Let N_t also be a random variable:

$$Y = \sum_{j=1}^{N_t} X_j \qquad X_j \overset{iid}{\sim} \Pr(x_1) \qquad N_t \sim \mathrm{Pois}(\lambda)$$

$$\begin{aligned}
\mathbb{C}_Y(\theta) &= \ln\left(\mathbb{E}_{\Pr(N_1, Y_1)}\left[\exp^{-\theta Y_1}\right]\right)\\
&= \ln\left(\mathbb{E}_{\Pr(N_1)}\left\{\mathbb{E}_{\Pr(X_1, \ldots X_{N_1})}\left[\exp^{-\theta\sum_{j=1}^{N_1} X_j} | N_1\right]\right\}\right)\\
&= \ln\left(\sum_{k=0}^{\infty}\left\{\int_{X_1, \ldots X_k} \exp^{-\theta\sum_{j=1}^{k} X_j} \Pr(x_1)\ldots\Pr(x_k)\right\}\Pr(N_1 = k)\right)\\
&= \ln\left(\sum_{k=0}^{\infty}\left\{\left[\int_{X_1}\exp^{-\theta X_j}\Pr(x_1)\right]^k\right\}\Pr(N_1 = k)\right)\\
&= \ln\left(\sum_{k=0}^{\infty}\left\{\left[\exp^{\mathbb{C}_\theta(x_1)}\right]^k\right\}\Pr(N_1 = k)\right)\\
&= \ln\left(\sum_{k=0}^{\infty}\exp^{k\mathbb{C}_\theta(x_1)}\Pr(N_1 = k)\right)\\
&= \ln\left(\mathbb{E}_{\Pr(N_1)}\left[N_1\mathbb{C}_{X_1}(\theta)\right]\right)\\
&= \mathbb{C}_{N_1}\left(-\mathbb{C}_{X_1}(\theta)\right) = -\lambda\left(1 - \exp^{\mathbb{C}_{X_1}(\theta)}\right) = \lambda\left(\exp^{\mathbb{C}_{X_1}(\theta)} - 1\right)
\end{aligned}$$

Compound Poisson Process (2)

Let $G_0(A) = \lambda$ $\qquad \pi_j, \pi \overset{iid}{\sim} \Pr(\pi)$

$$
\begin{aligned}
\mathbb{C}_Y(\theta) &= \lambda \left(\exp^{\mathbb{C}_\pi(\theta)} - 1 \right) \\
&= \lambda \left(\exp^{\ln\left[\int_{\pi \in \mathbb{R}} \exp^{-\theta\pi} \Pr(\pi) \Pr(d\pi)\right]} - 1 \right) \\
&= \int_{\omega \in A} G_0(\omega) \int_{\pi \in \mathbb{R}} \left(\exp^{-\theta \pi_j} \Pr(\pi) - 1 \right) \Pr(d\pi) \\
&= \int_{\omega \in A} \int_{\pi \in \mathbb{R}} \left(\exp^{-\theta \pi_j} \Pr(\pi) - 1 \right) \underbrace{G_0(\omega) \Pr(d\pi)}_{\nu(d\pi, d\omega)}
\end{aligned}
$$

For normal Poisson Process:

$$
\mathbb{C}_Y(\theta) = \int_{\omega \in A} \int_{\pi \in \mathbb{R}} \left(\exp^{-\theta \pi_j} \Pr(\pi) - 1 \right) \underbrace{G_0(\omega) \delta_\pi}_{\nu(d\pi, d\omega)}
$$

Gamma Process

- $G \sim \Gamma\mathrm{P}(\alpha, G_0), \qquad \forall A_1, \dots, A_K \in \Omega$:

$$G(A_i) \sim \mathrm{Gamma}(G_0(A_i), \alpha)$$

$$\text{where } \mathrm{Gamma}(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp^{-bx}$$

- Let $G = \{(\pi_i, \omega_i)\}_{i=1}^{\infty}$ be a realization in product space $\mathbb{R}^+ \times \Omega$:

$$\begin{aligned} G &\sim GaP(\alpha, G_0) \\ &= \sum \pi_i \delta_{\theta_i} \\ \text{where} \qquad \pi_i &\sim \underbrace{\pi^{-1} \exp^{-\alpha\pi}}_{\mathrm{Gamma}(0,\alpha)} \mathrm{d}\pi \\ \omega_i &\sim G_0 \end{aligned}$$

- is a Completely Random Measure with Levy intensity:

$$\nu(\mathrm{d}\pi, \mathrm{d}\omega) = \pi^{-1} \exp^{-\alpha\pi} \mathrm{d}\pi \, G_0(\mathrm{d}\omega)$$

Gamma Process: Derive its levy intensity (1)

Using defintion $\text{Gamma}(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp^{-bx} \implies G(A) \sim \text{Gamma}(G_0(A), \alpha)$

$$
\begin{aligned}
\phi_{G(A)}(\theta) &= \mathbb{E}_{\text{Gamma}(\underbrace{G(A)}_{\pi}, \underbrace{G_0(A)}_{a}, \underbrace{\alpha}_{b})} \left[\exp^{i\theta G(A)} \right] \\
&= \int_{\pi=0}^{\infty} \exp^{i\theta\pi} \, \text{Gamma}(\pi; a, b) = \int_{\pi=0}^{\infty} \exp^{i\theta\pi} \frac{b^a}{\Gamma(a)} \pi^{a-1} \exp^{-b\pi} \, d\pi \\
&= \int_{\pi=0}^{\infty} \frac{b^a}{\Gamma(a)} \pi^{a-1} \exp^{-\pi(b-i\theta)} \, d\pi
\end{aligned}
$$

Let $y = \pi(b - i\theta) \implies \pi = \frac{y}{(b-i\theta)}$ $\qquad \frac{dy}{d\pi} = (b - i\theta) \implies d\pi = \frac{dy}{(b-i\theta)}$

$$
\begin{aligned}
&= \frac{b^a}{\Gamma(a)} \int_{\pi=0}^{\infty} \left(\frac{y}{b - i\theta} \right)^{a-1} \exp^{-y} \frac{1}{(b - i\theta)} dy \\
&= \frac{b^a}{(b - i\theta)^a} \underbrace{\frac{1}{\Gamma(a)} \int_{\pi=0}^{\infty} y^{a-1} \exp^{-y} \, dy}_{=1} \\
&= \frac{b^a}{(b - i\theta)^a} = \left(\frac{b}{b - i\theta} \right)^a = \left(\frac{b - i\theta}{b} \right)^{-a} = \left(1 - \frac{-i\theta}{b} \right)^{-a}
\end{aligned}
$$

Gamma process: Derive its levy intensity (2)

- $\phi_{G(A)}(\theta) = \left(1 - \frac{-i\theta}{b}\right)^{-a}$ looks nothing like Levy-Khintchine representation
- From Frullani integral: if $f'(x)$ is continous and integral converges, $f(0)$, $f(\infty)$ are finite, then:

$$\int_{x=0}^{\infty} \frac{f(Mx) - f(Nx)}{x} \mathrm{d}x = [f(0) - f(\infty)] \ln\left(\frac{N}{M}\right)$$

$$\begin{aligned} f(Mx) = a \exp^{-(b-i\theta)x} &\implies M = (b - i\theta) \\ f(Nx) = a \exp^{-bx} &\implies N = b \\ &\implies f(x) = a \exp^{-x} \end{aligned}$$

$$[f(0) - f(\infty)] \ln\left(\frac{N}{M}\right) = a\left[\exp^{-0} - \exp^{-\infty}\right] \ln\left(\frac{b}{b - i\theta}\right) = a \ln\left(\frac{b}{b - i\theta}\right) = \ln\left[\left(1 - \frac{i\theta}{b}\right)^{-a}\right]$$

$$\begin{aligned} \left[\left(1 - \frac{i\theta}{b}\right)^{-a}\right] &= \exp\left[\int_{\pi=0}^{\infty} \frac{a \exp^{-(b-i\theta)\pi} - a \exp^{-b\pi}}{\pi} \mathrm{d}\pi\right] = \exp\left[a \int_{\pi=0}^{\infty} \frac{\exp^{-(b-i\theta)\pi} - \exp^{-b\pi}}{\pi} \mathrm{d}\pi\right] \\ &= \exp\left[a \int_{\pi=0}^{\infty} \left(\exp^{-i\theta r} - 1\right) \pi^{-1} \exp^{-b\pi} \mathrm{d}\pi\right] \\ &= \exp\left[G_0(A) \int_{\pi=0}^{\infty} \left(\exp^{-i\theta r} - 1\right) \pi^{-1} \exp^{-\alpha\pi} \mathrm{d}\pi\right] \\ &= \exp\left[\int_{\omega \in A} \int_{\pi=0}^{\infty} \left(\exp^{-i\theta r} - 1\right) \underbrace{\pi^{-1} \exp^{-\alpha\pi} \mathrm{d}\pi G_0(\mathrm{d}\omega)}_{\nu(\mathrm{d}\pi, \mathrm{d}\omega)}\right] \end{aligned}$$

Beta Process

- G is a Beta process, with base distribution G_0 and concentration parameter α:

$$
\begin{aligned}
G &\sim BP(\alpha G_0), \text{if} \\
G(A_k) &\sim \mathrm{Beta}(\alpha G_0(A_k), \alpha(1 - G_0(A_k)))
\end{aligned}
$$

- Given an infinitesimal partition $(A_1, \dots, A_K)$ with $K \to \infty$ and $G_0(A_k) \to 0$ the samples correspond to the density function:

$$
\begin{aligned}
& G = \sum \pi_i \delta_{\omega_i} \\
\text{where} \qquad & \pi_i \sim \mathrm{Beta}(0, \alpha) \\
& \omega_i \sim G_0
\end{aligned}
$$

- Beta process is a Completely Random Measure with Levy measure on product space $[0, 1] \times \Omega$ with Levy measure:

$$\nu(\mathrm{d}\pi d\theta) = \alpha\pi^{-1}(1 - \pi)^{\alpha-1}\mathrm{d}\pi G_0(\mathrm{d}\omega).$$

Total measure: $\nu^{+}(\mathbb{R}^{+}, \Omega)$

- Compound Poisson Process:

$$\nu(\mathrm{d}\pi, \mathrm{d}\omega) = G_0(\omega)\,\Pr(\mathrm{d}\pi) \implies \nu^{+}(\mathbb{R}^{+}, \Omega) = G_0(\Omega)$$

- Gamma Process:

$$\nu(\mathrm{d}\pi, \mathrm{d}\omega) = \pi^{-1} \exp^{-\alpha\pi} \mathrm{d}\pi\, G_0(\mathrm{d}\omega) \implies \nu^{+}(\mathbb{R}^{+}, \Omega) = \infty$$

- Beta Process:

$$\nu(\mathrm{d}\pi, \mathrm{d}\omega) = \alpha\pi^{-1}(1-\pi)^{\alpha-1} \mathrm{d}\pi\, G_0(\mathrm{d}\omega) \implies \nu^{+}(\mathbb{R}^{+}, \Omega) = \infty$$

Negative Binomial Process

- X is a Negative Binomial process, with base distribution G_0 and parameter p:

$$
\begin{aligned}
X &\sim \mathrm{NBP}(G_0, p), \text{ if} \\
X(A_k) &\sim \mathrm{NB}(G_0(A_k), p)
\end{aligned}
$$

Using identity $-\ln(1-p) = \sum_{n=1}^{\infty} \frac{p^n}{n}$

$$
\begin{aligned}
\mathbb{C}_{X(A)}(\theta) = \ln\left\{\mathbb{E}\left[\exp^{\theta X(A)}\right]\right\} &= G_0(A)\left[\ln(1-p) - \ln(1-p\exp^{i\theta})\right] \\
&= G_0(A)\left[\sum_{n=1}^{\infty} \frac{p^n \exp^{i\theta n}}{n} - \frac{p^n}{n}\right] = G_0(A)\left[\sum_{n=1}^{\infty}(\exp^{i\theta n} - 1)\frac{p^n}{n}\right] \\
&= G_0(A)\left[\sum_{n=1}^{\infty}(\exp^{i\theta n} - 1)\sum_{m=1}^{\infty} \delta_m(\mathrm{d}n)\frac{p^m}{m}\right] \\
&= \int_{\omega \in A} \sum_{n=1}^{\infty}(\exp^{i\theta n} - 1) \underbrace{\sum_{m=1}^{\infty} \delta_m(\mathrm{d}n)\frac{p^m}{m} G_0(\mathrm{d}\omega)}_{\nu(\mathrm{d}n, \mathrm{d}\omega)}
\end{aligned}
$$

Negative Binomial Process (2)

- Using identity $-\ln(1-p) = \sum_{n=1}^{\infty} \frac{p^n}{n}$

$$\mathbb{C}_{X(A)}(\theta) = \int_{\omega \in A} \sum_{n=1}^{\infty} (\exp^{i\theta n} - 1) \underbrace{\sum_{m=1}^{\infty} \delta_m(\mathrm{d}n) \frac{p^m}{m} G_0(\mathrm{d}\omega)}_{\nu(\mathrm{d}n, \mathrm{d}\omega)}$$

- **Total** measure: $\nu^+ = (\mathbb{Z}^+, \Omega) = -\ln(1-p) G_0(\Omega)$
- $\mathrm{Log}(x; p) = \frac{1}{-\ln(1-p)} \frac{p^x}{x}$
- a draw from the NBP consists of a **finite** number of distinct atoms almost surely

$$L \sim \mathrm{Pois}(\underbrace{-\ln(1-p) G_0(\Omega)}_{\nu^+}) \qquad X = \sum_{n=1}^{L} n_k \delta_{\omega_k}$$

$$n_k \sim \mathrm{Log}(n; p) \qquad \omega_k \sim \frac{G_0(\omega)}{G_0(\Omega)}$$

- However this is NOT conjugate

Negative Binomial Process - Chinese Restaurant Table Distribution

- $G \sim DP(\alpha, H)$ and N data points, the probability of K is:

$$\Pr(K = k|N, \alpha) = \mathrm{CRT}(K; N, \alpha) = \frac{\Gamma(\alpha)}{\Gamma(N+\alpha)} |s(N,k)| \alpha^k, \qquad k = 0, 1, \dots, N$$

- This means that,

$$\sum_{k=1}^{N} |s(N,k)| \alpha^k = \frac{\Gamma(N+\alpha)}{\Gamma(\alpha)}$$

- it can be sampled as $k = \sum_{n=1}^{N} b_n, \qquad b_n \sim \text{Bernoulli}\left(\frac{\alpha}{n-1+\alpha}\right)$

(From probability notes) Relationship between Multinomial distribution and Poisson

$$\text{Pois}(x|\lambda) = \frac{\lambda^x}{x!}\exp(-\lambda) \qquad \text{Mult}(n_1,\ldots,n_k|p_1,\ldots p_k) = \frac{(\sum n_i)!}{n_1!\ldots n_k!}\prod_{i=1}^k p_i^{n_i}$$

suppose:

- $x_1 \sim \text{Pois}(x|\lambda_1),\ldots,x_k \sim \text{Pois}(x|\lambda_k) \implies$
- The above generated two random variables:

1st random variable: $$\left(n = \sum_{i=1}^k x_i\right) \sim \text{Pois}(\lambda_1 + \lambda_2 + \cdots + \lambda_k)$$

2nd random variable: $$\mathbf{x} = (x_1,\ldots,x_k)|n \sim \text{Mult}(n, p_1,\ldots p_k) \text{ where } p_i = \frac{\lambda_i}{\sum_{j=1}^k \lambda_j}$$

Extend this Relationship to **Process**

- $X_1, \dots X_J$ for any measurable disjoint partition $A_1, \dots A_Q$ of Ω,
- Jointly model the count random variables $\{X_j(A_q)\}$.
- $X_j \sim \text{PoissonP}(G)$, with a shared Completely Random Measure G on Ω :

$$X_j(A) \sim \mathrm{Pois}(G(A))$$

$$\equiv X_j(\Omega) \sim \mathrm{Pois}(G(\Omega)) \qquad X_j | X_j(\Omega) \sim \text{MP}(X_j(\Omega), \tilde{G}) \qquad \text{where } \tilde{G} = \frac{G}{G(\Omega)}$$

$$\underbrace{X_j \sim \mathrm{NBP}\left(G_0, \frac{1}{c+1}\right) = \int_G \text{PP}(X_j|G)\text{GaP}(c, G_0)\text{d}G}_{\text{less preferred}}$$

$$\underbrace{\sim \mathrm{NBP}\left(G_0, p\right) = \int_G \text{PP}(X_j|G)\text{GaP}\left(\frac{J(1-p)}{p}, G_0\right)\text{d}G}_{\text{preferred}}$$

$$X = \left(\sum_{j=1}^{J} X_j\right) \sim \mathrm{NBP}(G_0, p) \qquad X_j(A) \sim \mathrm{NBP}(G_0(A), p)$$

Negative Binomial Process

- $\text{CRP}(K; N, \alpha) = \frac{\Gamma(\alpha)}{\Gamma(N+\alpha)} |s(N, k)| \alpha^k$
- $L \sim \mathrm{CRTP}(X, G_0)$ as CRT process:

$$\text{for each } A \in \Omega : \qquad L(A) = \sum_{\omega \in \Omega} L(\omega), \qquad L(\omega) \sim \mathrm{CRT}(\underbrace{X(\omega)}_{N}, \underbrace{G_0(\omega)}_{\alpha})$$

- $X(A)$ customer count and $L(A)$ table count. Each $A \in \Omega$. Number of tables:

$$L(A) \sim \text{Pois}(-G_0(A) \ln(1-p))$$

- assign Log(p) customers to each table, with $X(A)$ total number of customers.
- $X(A) \sim \text{NB}(G_0(A), p)$ customers and assign them into $L(A) \sim \sum_{\omega \in A} \mathrm{CRT}(X(\omega), G_0 \omega))$ tables:

$$\underbrace{L | G_0, p \sim \text{PoissonP}(-G_0 \ln(1-p))}_{\text{num of tables}} \qquad X | L, p \sim \sum_{t=1}^{L} \text{Log}(p)$$

$$\text{is equivalent:} \qquad \underbrace{X | G_0, p \sim \mathrm{NBP}(G_0, p)}_{\text{num of people}} \qquad L | X, G_0 \sim \mathrm{CRTP}(X, G_0)$$

Negative Binomial Process (2)

- **conjugate property**:

$$\text{Pois}(\{x_i\};\lambda)\quad \underbrace{\text{Gamma}(\lambda; a, b)}_{\frac{1}{\Gamma(k)\theta^k} x^{k-1}\exp^{-\frac{x}{\theta}}}$$

$$\propto \text{Gamma}\left(\lambda; k + \sum_{i=1}^{n} x_i, \frac{\theta}{n\theta + 1}\right)$$

$$\text{NB}(\{x_i\}; r, p)\text{Beta}(p; a, b)$$

$$\propto \text{Beta}\left(p; a + \sum_{i=1}^{N} x_i, b + rN\right)$$

- **prior**:

$$p \sim \text{Beta}(a_0, b_0)$$

- $G \sim \Gamma\text{P}(\alpha, G_0) \implies G(A_i) \sim \text{Gamma}(G_0(A_i), 1/\alpha)$
- Note the “flip” between α from process to distribution

- **posterior**:

$$\begin{aligned}
G|X_{1:J}, p, G_0 &= \underbrace{\text{PoissonP}(X; G)}_{\Pr(X|G)}\underbrace{\Gamma\text{P}\left(G; \frac{J(1-p)}{p}, G_0\right)}_{\Pr(G|G_0,p)} \\
&= \text{PoissonP}(X; G)\text{Gamma}\left(G; G_0, \frac{p}{J(1-p)}\right) \\
&= \text{Gamma}\left(G; G_0 + X, \frac{\frac{p}{J(1-p)}}{\frac{p}{J(1-p)} + 1}\right) \\
&= \Gamma\text{P}\left(\frac{\frac{p}{J(1-p)} + 1}{\frac{p}{J(1-p)}}, G_0 + X\right) \\
&= \Gamma\text{P}\left(1 + \frac{J(1-p)}{p}, G_0 + X\right) \\
&= \Gamma\text{P}\left(\frac{J}{p}, G_0 + X\right)
\end{aligned}$$

$$\begin{aligned}
p|X(\Omega), G_0(\Omega) &= \underbrace{\text{NB}(X(\Omega); G_0(\Omega), p)}_{\Pr(X(\Omega)|G_0(\Omega),p)}\underbrace{\text{Beta}(p; a_0, b_0)}_{p(p|a_0,b_0)} \\
&= \text{Beta}(p; a_0 + X(\Omega), b_0 + G_0(\Omega)) \\
&= \text{Beta}(p; a_0 + X(\Omega), b_0 + \gamma_0)
\end{aligned}$$

Negative Binomial Process (3)

- **conjugate property**:

$$\text{Pois}(\{x_i\};\lambda)\underbrace{\text{Gamma}(\lambda;a,b)}_{\frac{1}{\Gamma(k)\theta^k}x^{k-1}\exp{-\frac{x}{\theta}}}$$

$$\propto \text{Gamma}\left(\lambda; k+\sum_{i=1}^{n} x_i, \frac{\theta}{n\theta+1}\right)$$

- **prior**:

$$(\gamma_0 = G_0(\Omega)) \sim \text{Gamma}(e_0, 1/f_0)$$

- $X \sim \text{Gamma}(k,\theta) \implies$ $cX \sim \text{Gamma}(k, c\theta)$

- **posterior**:

$$L|X, G_0 \sim \text{CRTP}(X, G_0)$$

$$\begin{aligned}
\gamma_0|L(\Omega),p &= \underbrace{\text{Pois}(L(\Omega);-\gamma_0\ln(1-p))}_{\Pr(L|\gamma_0,p)}\underbrace{\text{Gamma}(\gamma_0;e_0,1/f_0)}_{\Pr(\gamma_0)}\\
&= \text{Pois}\left(L(\Omega);-\gamma_0\ln(1-p)\right)\\
&\qquad \text{Gamma}\left(-\gamma_0\ln(1-p); e_0, \frac{-\ln(1-p)}{f_0}\right)\\
&= \text{Gamma}\left(-\gamma_0\ln(1-p); e_0+L(\Omega), \frac{\frac{-\ln(1-p)}{f_0}}{\frac{-\ln(1-p)}{f_0}+1}\right)\\
&= \text{Gamma}\left(-\gamma_0\ln(1-p); e_0+L(\Omega), \frac{-\ln(1-p)}{-\ln(1-p)+f_0}\right)\\
&= \text{Gamma}\left(\gamma_0; e_0+L(\Omega), \frac{1}{-\ln(1-p)+f_0}\right)
\end{aligned}$$

A gamma-NB distribution:

$$m \sim \mathrm{NB}(r, p) \qquad r \sim \mathrm{Gamma}(r_1, 1/c_1)$$

$$m = \sum_{t=1}^{l} \mathrm{Log}(p) \qquad \underbrace{l \sim \mathrm{Pois}(-r \ln(1-p)) \qquad r \sim \mathrm{Gamma}(r_1, 1/c_1)}$$

$$= \sum_{t=1}^{l} \mathrm{Log}(p) \qquad l \sim \Pr(p, c_1) = \int_r \mathrm{Pois}(-r \ln(1-p))\mathrm{Gamma}(r_1, 1/c_1)\mathrm{d}r = \mathrm{NB}(r_1, p')$$

denote $p' = \frac{-ln(1-p)}{c_1 - ln(1-p)}$:

$$m \sim \sum_{t=1}^{l} \mathrm{Log}(p) \qquad l \sim \sum_{t'=1}^{l'} \mathrm{Log}(p') \qquad l' \sim \mathrm{Pois}(-r_1 \ln(1-p'))$$

Equivalently,

$$m \sim \sum_{t=1}^{l} \mathrm{Log}(p) \qquad l' \sim \mathrm{CRT}(l, r_1) \qquad l \sim \mathrm{NB}(r_1, p).$$

Sample Negative Binomial **distribution** - with a Gamma Prior

Let $\{m_{j1}, \ldots m_{jN_j}\}_{j=1,J}$:

$$\begin{aligned} m_{ji} &\sim \mathrm{NB}(r_j, p_j) \\ p_j &\sim \mathrm{Beta}(a_0, b_0) \\ r_j &\sim \mathrm{Gamma}(r_1, 1/c) \\ r_1 &\sim \mathrm{Gamma}(e_0, 1/f_0) \end{aligned}$$

$$m \sim \mathrm{NB}(r, p) \implies$$

$$m \sim \sum_{t=1}^{l} \mathrm{Log}(p), l \sim \mathrm{Pois}(-r \ln(1-p))$$

$$\implies l|m, r \sim \mathrm{CRT}(m, r)$$

$$\begin{aligned} l_{ji}|- &\sim \mathrm{CRT}(m_{ji}, r_j) \\ l'_j|- &\sim \mathrm{CRT}\left(\sum_{i=1}^{N_j} l_{ji}, r_1\right) \\ p'_j &= \frac{-N_j \ln(1-p_j)}{c - N_j \ln(1-p_j)} \\ r_1|- &\propto \prod_j^J \mathrm{NB}(l_j; r_1, p')\mathrm{Gamma}(r_1; e_0, 1/f_0) \\ &\sim \mathrm{Gamma}\left(e_0 + \sum_{j=1}^{J} l'_j, \frac{1}{f_0 - \sum_{j=1}^{J} \ln(1-p'_j)}\right) \\ r_j|- &\propto \prod_i^{N_j} \mathrm{NB}(m_{ji}; r_j, p_j)\mathrm{Gamma}(r_j; r_1, 1/c) \\ &\sim \mathrm{Gamma}\left(r_1 + \sum_{i=1}^{N_j} l_{ji}, \frac{1}{c - N_j \ln(1-p_j)}\right) \\ p_j|- &\sim \mathrm{Beta}\left(a_0 + \sum_{i=1}^{N_j} m_{ji}, b_0 + N_j r_j\right) \end{aligned}$$

Approximating $G \sim \Gamma\mathrm{P}(c, G_0)$

$$
\begin{aligned}
G_0 &= \sum_{k=1}^{K} \frac{\gamma_0}{K} \delta_{\omega_k} \\
G &= \sum_{k=1}^{K} r_k \delta_{\omega_k} \\
r_k &\sim \mathrm{Gamma}(\gamma_0/K, 1/c) \\
\omega_k &\sim g_0(\omega_k)
\end{aligned}
$$

$G \sim \Gamma\mathrm{P}(c, G_0)$ becomes a draw from the gamma process with a continuous base measure as $\to \infty$

Block Sampling Trick

$$m \sim \mathrm{NB}(r, p) \implies$$

$$m \sim \sum_{t=1}^{l} \mathrm{Log}(p) \qquad l \sim \mathrm{Pois}(-r \ln(1-p))$$

$$\implies l | m, r \sim \mathrm{CRT}(m, r)$$

$$n_{jk} | \gamma_0, p \sim \mathrm{NB}(\gamma_0/K, p) \implies$$

$$n_{jk} | l_k, p \sim \sum_{t=1}^{l_k} \mathrm{Log}(p) \qquad l_k | \gamma_0, p \sim \mathrm{Pois}(-\gamma_0/K \ln(1-p))$$

$$\implies l_k | n_{jk}, \gamma_0 \sim \mathrm{CRT}(n_{jk}, \gamma_0/K)$$

Block sample $r_k, n_{jk}, l_k \gamma_0 | p$:

$$\begin{aligned}
\Pr(n_{jk} | r_k, l_k, \gamma_0, p) &= \Pr(n_{jk} | r_k) \\
&= \mathrm{Pois}(r_k) \\
p(r_k | n_{jk}, l_k, \gamma_0, p) &= \Pr(n_{jk} | r_k) p(r_k | \gamma_0, p) \\
&= \mathrm{Pois}(n_{jk}) \mathrm{Gamma}(r_k; \gamma_0/K, p/(1-p)) \\
\Pr(l_k | n_{jk}, r_k, \gamma_0, p) &= \Pr(l_k | n_{jk}, \gamma_0) \\
&= \mathrm{CRT}(l_k; n_{jk}, \gamma_0/K) \\
\Pr(\gamma_0 | \{l_k\}, n_{jk}, r_k, p) &= \Pr(\{l_k\} | n_{jk}, \gamma_0) p(\gamma_0) \\
&= \prod_{k=1}^{K} \mathrm{Pois}(l_k; -\gamma_0/K \ln(1-p)) \mathrm{Gamma}(e_0, 1/f_0) \\
&= \Pr(n_{jk} | \gamma_0, p) p(\gamma_0) \\
&= \underbrace{\mathrm{NB}(n_{jk}; \gamma_0/K, p) \mathrm{Gamma}(\gamma_0; e_0, 1/f_0)}_{\text{not conjugate}}
\end{aligned}$$

Let $G_0 = \sum_{k=1}^{K} \frac{\gamma_0}{K} \delta_{\omega_k}$:

$$
\begin{aligned}
x_{ji} &\sim F(\omega_{z_{ji}}) \\
\omega_k &\sim g_0(\omega_k) \\
&\underbrace{n_{jk} \sim \mathrm{NB}(\gamma_0/K, p)}_{n_{jk} \sim \mathrm{Pois}(r_k) \quad r_k \sim \mathrm{Gamma}(\gamma_0/K, p/(1-p))} \\
N_j &= \sum_{k=1}^{K} n_{jk} \\
p &\sim \mathrm{Beta}(a_0, b_0) \\
\gamma_0 &\sim \mathrm{Gamma}(e_0, 1/f_0)
\end{aligned}
$$

$$
\begin{aligned}
\Pr(z_{ji} = k|-) &\propto F(x_{ji}; \omega_k) r_k \\
l_k|- &\sim \mathrm{CRT}(n_k, \gamma_0/K) \\
\gamma_0|- &\propto \prod_k^K \mathrm{NB}(l_k; n_k, \gamma_0/K)\mathrm{Gamma}(\gamma_0; e_0, 1/f_0) \\
&\sim \mathrm{Gamma}\left(e_0 + \sum_{k=1}^{K} l_k, \frac{1}{f_0 - \ln(1-p)}\right) \\
p|- &\sim \mathrm{Beta}\left(a_0 + \sum_{i=1}^{J} N_j, b_0 + \gamma_0\right) \\
r_k|- &\sim \mathrm{Gamma}(\gamma_0/K + n_k, p/J) \\
p(\omega_k|- &\propto \prod_{z_{ji}=k} F(x_{ji}; \omega_k) g_0(\omega_k)
\end{aligned}
$$

$$
\underbrace{N_j \sim \mathrm{Pois}(r) \qquad (n_{j1}, \dots, n_{jK}) \sim \mathrm{Mult}(N_j; r_1/r, \dots, r_K/r)}_{z_{ji} \sim \mathrm{Discrete}(r_1/r, \dots, r_K/r) \qquad n_{jk} = \sum \delta(z_{ji} = k)}
$$

Gamma Negative Binomial Process

Let $G_0 = \sum_{k=1}^{K} \frac{\gamma_0}{K} \delta_{\omega_k}$:

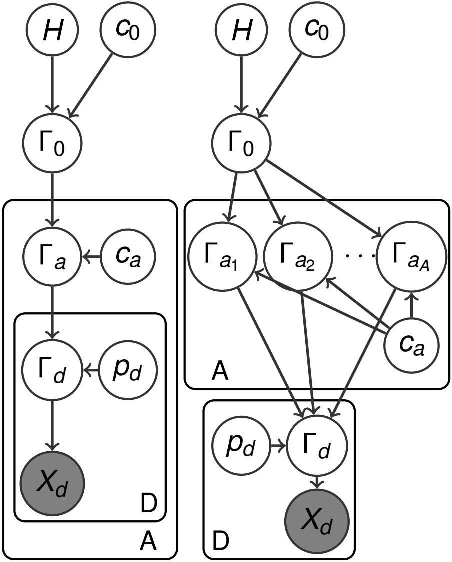
$$
\begin{aligned}
x_{ji} &\sim F(\omega_{z_{ji}}) \\
\omega_k &\sim g_0(\omega_k) \\
r_k &\sim \mathrm{Gamma}(\gamma_0/K, 1/c) \\
&\underbrace{n_{jk} \sim \mathrm{NB}(r_k, p_j)}_{n_{jk} \sim \mathrm{Pois}(\theta_{jk}) \quad \theta_{jk} \sim \mathrm{Gamma}(r_k, p_j/(1-p_j))} \\
N_j &= \sum_{k=1}^{K} n_{jk} \\
p_j &\sim \mathrm{Beta}(a_0, b_0) \\
\gamma_0 &\sim \mathrm{Gamma}(e_0, 1/f_0)
\end{aligned}
$$

$$
\begin{aligned}
\Pr(z_{ji} = k | -) &\propto F(x_{ji}; \omega_k)\theta_{jk} \\
l_{jk} | - &\sim \mathrm{CRT}(n_{jk}, r_k) \\
l'_k | - &\sim \mathrm{CRT}\left(\sum_j l_{jk}, \gamma_0/K\right) \\
p_j | - &\sim \mathrm{Beta}(a_0 + N_j, b_0 + r_k) \\
p' &= \frac{-\sum_j \ln(1-p_j)}{c - \sum_j \ln(1-p_j)} \\
\gamma_0 | - &\sim \mathrm{Gamma}\left(e_0 + \sum_k l'_k, \frac{1}{f_0 - \ln(1-p')}\right) \\
\gamma_k | - &\sim \mathrm{Gamma}\left(\gamma_0/K + \sum_j l_{jk}, \frac{1}{c - \ln(1-p_j)}\right) \\
\theta_{jk} | - &\sim \mathrm{Gamma}(r_k + n_{jk}, p_j) \\
p(\omega_k | - &\propto \prod_{z_{ji}=k} F(x_{ji}; \omega_k) g_0(\omega_k)
\end{aligned}
$$

Block sample $r_k, \theta_k, n_{jk}, l_k, l'_k, \gamma_0 | p$:

$$
\begin{aligned}
\Pr(n_{jk}|r_k, \theta_k, l_k, l'_k, \gamma_0, p) &= \Pr(n_{jk}|\theta_k) = \mathrm{Pois}(n_{jk}; \theta_k) \\
p(r_k|n_{jk}, l_k, \gamma_0, p) &= \Pr(n_{jk}|r_k) p(r_k|\gamma_0, p) \\
&= \mathrm{Pois}(n_{jk}) \mathrm{Gamma}(r_k; \gamma_0/K, p/(1-p)) \\
\Pr(l_k|n_{jk}, r_k, \gamma_0, p) &= \Pr(l_k|n_{jk}, \gamma_0) = \mathrm{CRT}(l_k; n_{jk}, \gamma_0/K) \\
\Pr(\gamma_0|\{l_k\}, n_{jk}, r_k, p) &= \Pr(\{l_k\}|n_{jk}, \gamma_0) p(\gamma_0) \\
&= \prod_{k=1}^{K} \mathrm{Pois}(l_k; -\gamma_0/K \ln(1-p)) \mathrm{Gamma}(e_0, 1/f_0) \\
&= \Pr(n_{jk}|\gamma_0, p) p(\gamma_0) \\
&= \underbrace{\mathrm{NB}(n_{jk}; \gamma_0/K, p) \mathrm{Gamma}(\gamma_0; e_0, 1/f_0)}_{\text{not conjugate}}
\end{aligned}
$$

Multiple Author Model



$$\Gamma_0 \sim \Gamma\mathrm{P}(c_0, H)$$
$$\Gamma_a \sim \Gamma\mathrm{P}(c_a, \Gamma_0)$$
$$\Gamma_a^d = \Gamma_{a_1} \oplus \Gamma_{a_2} \oplus \cdots \oplus \Gamma_{a_{A_d}}$$
$$X_d \sim \mathrm{NBP}\left(\Gamma_a^d, p\right)$$

$$\Gamma_0 \sim \Gamma\mathrm{P}(c_0, H)$$
$$\Gamma_a \sim \Gamma\mathrm{P}(c_a, \Gamma_0)$$
$$\Gamma_a^d = \Gamma_{a_1} \oplus \Gamma_{a_2} \oplus \cdots \oplus \Gamma_{a_{A_d}}$$
$$\Gamma_d \sim \Gamma\mathrm{P}\left(\frac{1-p_d}{p_d}, \Gamma_a^d\right)$$
$$X_d \sim \mathrm{PoissonP}(\Gamma_d)$$

Multiple Author Model

$$
\begin{aligned}
\gamma_0 &\sim \text{Gamma}(e_0, 1/f_0) \\
r_{0,k}|\gamma_0, c_0 &\sim \text{Gamma}(\gamma_0/K, 1/c_0) \\
r_{a,k}|r_0, c_a &\sim \text{Gamma}(r_{0,k}, 1/c_a) \\
p_d &\sim \text{beta}(a_0, b_0) \\
r^d_{a,k} &= r_{a_1,k} \oplus r_{a_2,k} \oplus \cdots \\
r_{d,k}|r_a, p_d &\sim \text{Gamma}\left(r^d_{a,k}, \frac{p_d}{1-p_d}\right) \\
n_{d,k} &\sim \text{Pois}(r_{d,k}) \\
N_d &= \sum_{k=1}^{K} n_{d,k} \\
\theta_{1:K} &\sim \frac{1}{\gamma_0} H \\
z_{d,n} &\sim \text{Mult}\left(\frac{r_{d,1}}{\sum r_d}, \frac{r_{d,2}}{\sum r_d}, \frac{r_{d,3}}{\sum r_d}, \cdots\right) \\
w_{d,n} &\sim \theta_{z_{d,n}}
\end{aligned}
$$

where $\gamma_0 = \int dH$

$$
\begin{aligned}
\gamma_0 &\sim \text{Gamma}(e_0, 1/f_0) \\
r_{0,k}|\gamma_0, c_0 &\sim \text{Gamma}(\gamma_0/K, 1/c_0) \\
r_{a,k}|r_0, c_a &\sim \text{Gamma}(r_{0,k}, 1/c_a) \\
p_d &\sim \text{beta}(a_{d,0}, b_{d,0}) \\
r^d_{a,k} &= r_{a_1,k} \oplus r_{a_2,k} \oplus \cdots \\
r_{d,k}|r_a, p_d &\sim \text{Gamma}\left(r^d_{a,k}, \frac{p_d}{1-p_d}\right) \\
r^a_{d,k} &\sim \text{Gamma}\left(\frac{r_{a,k}}{A_d}, \frac{p_d}{1-p_d}\right), \quad a \in A^d \\
z^a_{d,n} &\sim \text{Discrete}\left(\frac{r^a_{d,k}}{r}, \cdots\right) \\
N_d &= \sum_n \sum_a z^a_{d,n} \\
n_{d,k} &= \sum_n \delta(z_{d,n} = k) \\
n_{a,k} &= \sum_d \sum_n \delta(z_{d,n} = k \text{ AND } i_{d,n} = a) \\
n^a_{d,k} &= \sum_n \delta(z_{d,n} = k \text{ AND } i_{d,n} = a)
\end{aligned}
\tag{1}
$$