Sequential Monte Carlo: Particle Filter

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Importance sampling again

To approximate the integral, but $p(z)$ is hard to sample.

$$\begin{aligned}
\mathrm{E}_{p(z)}[f(z)] &= \int f(z)p(z)dz \\
&= \int \underbrace{f(z)\frac{p(z)}{q(z)}}_{\text{new}\tilde{f}(z)} q(z)dz \\
&\approx \frac{1}{N}\sum_{n=1}^{N} f(z^i)\frac{p(z^i)}{q(z^i)}
\end{aligned} \tag{1}$$

Revision on SMC

Take Importance Sampling to higher dimensions, the importance weights are:

$$w_n(x_{1:n}) = \frac{\gamma(x_{1:n})}{q_n(x_{1:n})} \tag{2}$$

Hard to choose $q(.)$ in high-dimension
Solution : rewrite equation (2) in the following:

$$w_n(x_{1:n}) = \frac{\gamma(x_{1:n})}{q_n(x_n|x_{1:n-1})q_{n-1}(x_{1:n-1})} \times \frac{\gamma(x_{1:n-1})}{\gamma(x_{1:n-1})}$$

re-arrange:

$$w_n(x_{1:n}) = \frac{\gamma(x_{1:n})}{q(x_{1:n})} = w_{n-1}(x_{1:n-1}) \times \frac{\gamma(x_{1:n})}{\gamma(x_{1:n-1})q(x_n|x_{1:n-1})}$$

Revision on SMC (2)

Top-down:

$$w_n(x_{1:n}) = w_{n-1}(x_{1:n-1}) \frac{\gamma(x_{1:n})}{\gamma(x_{1:n-1}) q(x_n | x_{1:n-1})} \tag{3}$$

Bottom-up:

$$w_n(x_{1:n}) = w_1(x_1) \prod_{j=2}^{n} \frac{\gamma(x_{1:j})}{\gamma(x_{1:j-1}) q(x_j | x_{1:j-1})}$$

The two are equivalent

Just too easy to put it all in an algorithm:

The SIS algorithm:

$$
\begin{aligned}
&\text{At dimension } n = 1 \text{: For each particle } i\\
&\quad \text{Sample } x_1^i \sim q_1(x_1)\\
&\quad \text{Compute the weights } w_1^i \propto \frac{\gamma(x_1^i)}{q_1(x_1^i)}\\
&\text{At dimension } n \geq 2 \text{: For each particle } i\\
&\quad \text{Sample } x_n^i \sim q_n(x_n | x_{1:n-1}^i)\\
&\quad \text{Compute the weights } w_n^i \propto w_{n-1}^i \frac{\gamma(x_{1:n}^i)}{\gamma(x_{1:n-1}^i) q(x_n^i | x_{1:n-1}^i)}
\end{aligned}
\tag{4}
$$

Particle Filter

Put this in a state-space setting, you have particle filter!
By changing n to t to reflect time sequentiality. In here, we assume that:

$$p(x_{1:t}|y_{1:t}) = \frac{p(x_{1:t}, y_{1:t})}{p(y_{1:t})} = \frac{\gamma_t(x_{1:t})}{\mathcal{Z}}$$

In here, we assume:

$$\begin{aligned}
\gamma_t(x_{1:t}) &= p(x_{1:t}, y_{1:t}) \\
&= p(y_t|x_{1:t}, y_{1:t-1})p(x_t|x_{1:t-1}, y_{1:t-1})\gamma_{t-1}(x_{1:t-1}) \\
&= p(y_t|x_t)p(x_t|x_{t-1})\gamma_{t-1}(x_{1:t-1})
\end{aligned} \tag{5}$$

Particle Filter

Divide by the proposal distribution $q(.)$, and do the same trick, this time, we use:

$$w_t(x_{1:t}) = \frac{\gamma_{(1:t)}}{q_{(1:t)}} = \frac{\gamma_{(1:t-1)}}{q_{(1:t-1)}} \times \frac{p(y_t|x_t)p(x_t|x_{t-1})}{q(x_t|x_{1:t-1})}$$

we can make a "reasonable" assumption that:

$$q(x_t|x_{1:t-1}) \equiv q(x_t|x_{t-1}, y_t) \tag{6}$$

Hence,

$$w_t(x_{1:t}) = w_{t-1}(x_{1:t-1}) \times \frac{p(y_t|x_t)p(x_t|x_{t-1})}{q(x_t|x_{t-1}, y_t)}$$

question is How are we going to choose $q(.)$ **a short answer** Choose $q(.)$ somehow from your dynamic model

Optimal proposal: $q(x_t|x_{k-1}, y_t) = p(x_t|x_{k-1}, y_t)$

Stated in [Doucet 1998], $q(x_t|x_{k-1}, y_t) = p(x_t|x_{k-1}, y_t)$ is optimal,then:

$$
\begin{aligned}
w_{(1:t)} &\propto w_{(1:t-1)} \times \frac{p(y_t|x_t)p(x_t|x_{t-1})}{p(x_t|x_{t-1}, y_t)} \\
&= w_{(1:t-1)} \times \frac{p(y_t|x_t)p(x_t|x_{t-1})p(y_t|x_{t-1})p(x_{t-1})}{p(y_t|x_t)p(x_t|x_{t-1})p(x_{t-1})} \\
&= w_{(1:t-1)} \times p(y_t|x_{t-1})
\end{aligned}
$$

However, $p(y_t|x_{t-1})$ is quite meaningless:

$$
w_{(1:t)} \propto w_{(1:t-1)} \times \int_{x_t} p(y_t|x_t)p(x_t|x_{t-1}) \tag{7}
$$

Two problem: (1) Difficult to sample from $p(x_t|x_{k-1}, y_t)$ and (2) integral is difficult to perform!

Main talk: sub-optimal methods

In this talk, I will present two "popular" sub-optimal sampling methods first:

- Bootstrap Particle Filter
- Auxiliary Particle Filter

Bootstrap Particle Filter

Sometimes calling it Condensational Filter. (Famous Michael Isard)
Let $q(x_t|x_{k-1}, y_t) = p(x_t|x_{k-1})$, i.e., y_t does not participate in the proposal $q(.)$

$$
\begin{aligned}
w_{(1:t)} &\propto w_{(1:t-1)} \times \frac{p(y_t|x_t)p(x_t|x_{t-1})}{p(x_t|x_{t-1})} \\
&= w_{(1:t-1)} \times p(y_t|x_t)
\end{aligned} \tag{8}
$$

- particles x_t^i are sampled from $p(.|x_{t-1})$, but are weighted by $p(y_t|x_t^i)$
- the danger is that x_t^i may receive close to zero weight if $p(y_t|x_t^i)$ is very small.

The Condensational Filter algorithm:

$$
\begin{aligned}
&\text{At time } t \\
&\text{For each particle } i: \\
&\quad \text{Sample } x_t^i \sim p(x_t | x_{t-1}^i) \qquad \left(\text{Or } x_1^i \sim p(x_1) \text{ when } t = 1\right) \\
&\quad \text{Compute the weights } w_t^i \propto \pi_{t-1}^i p(y_t | x_t^i) \\
&\text{normalize weights } \pi_t^i = \frac{w_t^i}{\sum_{i=1}^N w_t^i}
\end{aligned}
\tag{9}
$$

Problem particle degeneracy occurs very quickly.
Solution break those big particle into smaller ones, from the "re-sampling" step. To determine if "big particles" exist, check effective particle size.
BTW re-sampling does not solve particle degeneracy problem altogether.

Introducing Re-Sampling

Re-sampling sometimes can be considered as jointly “sample” an index i^j to indicate which $x_{t-1}^{i^j}$ generated x_t^i, and x_t^i itself.

$$\begin{aligned} x_t^i &\sim q(x_t | x_{t-1}^i, y_t) \\ &\text{becomes:} \\ j &\sim \pi_{t-1}(x_{1:t-1}) \\ x_t^i &\sim q(x_t | x_{t-1}^{i^j}, y_t) \end{aligned} \tag{10}$$

For each particle i at time t, you get (x_t^i, i^j).

Introducing Re-Sampling

Substituting N of the (x_t^i, i^j) into the following:

$$w_t(x_{1:t}) \propto \pi_{t-1}(x_{1:t-1}) \times \frac{p(y_t|x_t)p(x_t|x_{t-1})}{q(x_t|x_{t-1}, y_t)}$$

$$\begin{aligned} w_t^i(x_{1:t}) &\propto \pi^{i^j}_{(t-1)} \times \frac{p(y_t|x_t^i)p(x_t^i|x_{t-1}^{i^j})}{\pi^{i^j}_{(t-1)} q(x_t^i|x_{t-1}^{i^j}, y_t)} \\ &= \frac{p(y_t|x_t^i)p(x_t^i|x_{t-1}^{i^j})}{q(x_t^i|x_{t-1}^{i^j}, y_t)} \end{aligned}$$

In the bootstrap filter:

$$w_t^i(x_{1:t}) \propto p(y_t|x_t^i)$$

The Condensational Filter algorithm:

At time t

For each i:

$$\begin{aligned}
&\text{Sample } j \sim \pi_{t-1}(x_{1:t-1}) \qquad - \text{choose an an ancestor} \\
&\text{Sample } x_t^i \sim p(x_t | x_{t-1}^{i^j}) \qquad \big(\text{Or } x_1^i \sim p(x_1) \text{ when } t = 1\big) \\
&\text{Compute the weights } w_t^i \propto p(y_t | x_t^i)
\end{aligned} \tag{11}$$

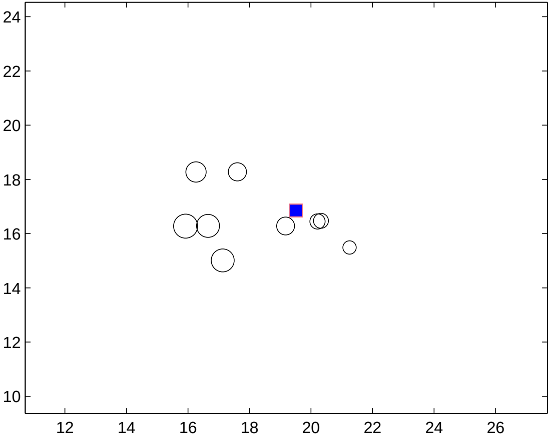
normalize weights $\pi_t^i = \dfrac{w_t^i}{\sum_{i=1}^N w_t^i}$

A little demo

- $p(x_t|x_{t-1}) = \mathcal{N}(Ax_{t-1} + B, Q)$
- $p(y_t|x_t) = \mathcal{N}(x_t, R)$

This is just for demo purpose, you can compute $p(x_t|y_{1:t})$ exactly using Kalman Filter!

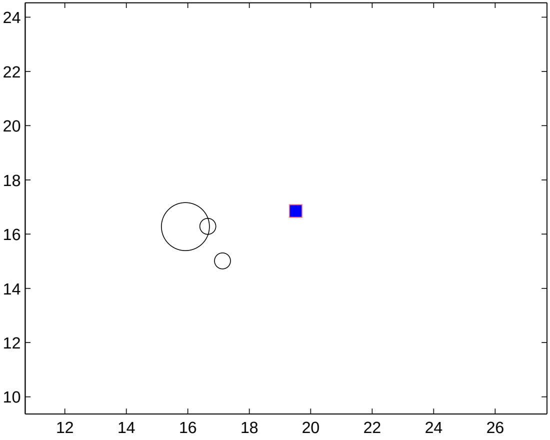
Representation for $p(x_{t-1}|y_{1:t-1})$



- Circles are weighted particle representation of $p(x_{t-1}|y_{1:t-1})$
- The blue square is y_t

Re-sampling

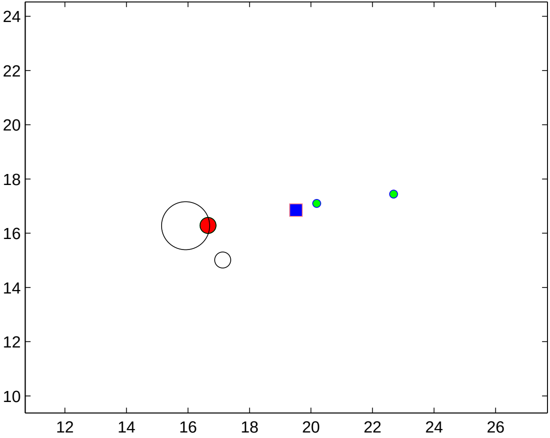
To sample $j \sim \pi_{t-1}(x_{1:t-1})$:



- Size of the circle indicates the number of times $x_{t-1}^{i^j}$ was selected.

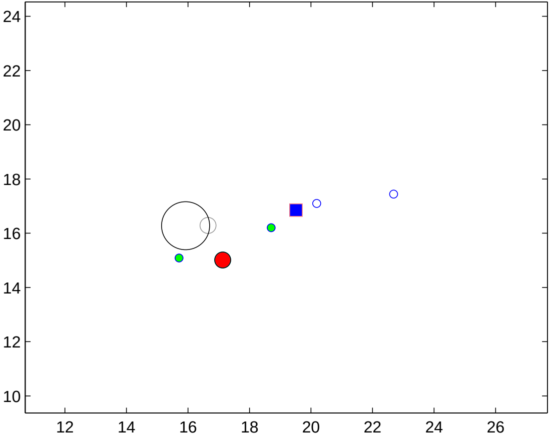
Transition demos

Sample $x_t^i \sim p(x_t | x_{t-1}^{i^j}) : \forall i^j = 1$



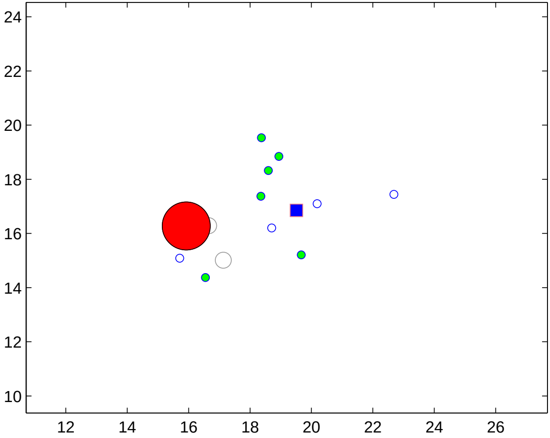
Transition demos

Sample $x_t^i \sim p(x_t | x_{t-1}^{i^j}) : \forall i^j = 2$



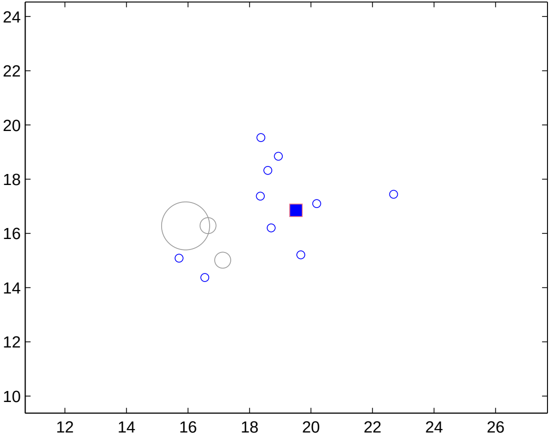
Transition demos

Sample $x_t^i \sim p(x_t | x_{t-1}^{i^j}) : \forall i^j = 3$



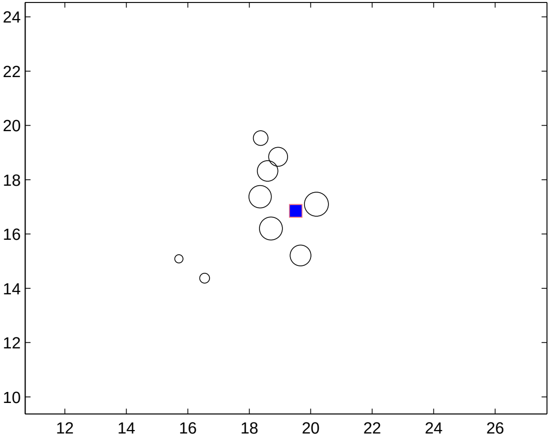
Transition demos

Here are the complete $\{x_t^i\}_1^N$ sampled.



After re-weighting

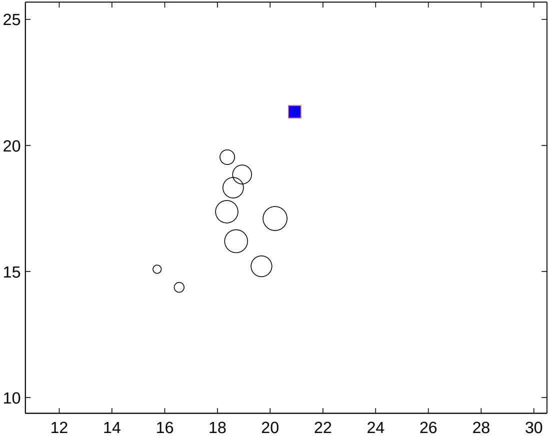
Compute the weights $w_t^i \propto p(y_t|x_t^i)$:



The above is the representation for $p(x_t|y_{1:t})$ Note that weights are in log scale

Next t

So the recursion will repeat:



The above is the representation for $p(x_{t-1}|y_{1:t-1})$ in the next t:

Some cool things you can do just with Bootstrap Filter

For example, A Coupled two-states dynamic model:
To estimate $p(x^1_{1:t}, x^2_{1:t}|y^1_{1:t}, y^2_{1:t})$

$$
\begin{aligned}
& w^i_t(x^1_{1:t}, x^2_{1:t}) \propto \\
= & \frac{g_1(y^1_t|x^1_t)g_2(y^2_t|x^2_t)f_1(x^1_t|x^1_{t-1}, x^2_{t-1})f_2(x^2_t|x^1_{t-1}, x^2_{t-1})}{q^1(x^1_t|y^1_t, x^1_{t-1}, x^2_{t-1})q^2(x^2_t|y^2_t, x^1_{t-1}, x^2_{t-1})} \\
& w^i_{t-1}(x^1_{1:t-1}, x^2_{1:t-1})
\end{aligned} \tag{12}
$$

Sampler for Coupled dynamic model

(leaving out the case of $t = 1$, and re-sampling step)

$$
\begin{aligned}
&\text{At time } t\text{:} \\
&\text{Sample } x_t^{1,(i)} \sim f_1(x_t^1 | x_{t-1}^{1,(i)}, x_{t-1}^{2,(i)}) \\
&\text{Sample } x_t^{2,(i)} \sim f_2(x_t^2 | x_{t-1}^{1,(i)}, x_{t-1}^{2,(i)}) \\
&\text{Compute the weights } w_t^{1,(i)} \propto \pi_{t-1}^{1,(i)} g_1(y_t^{1,(i)} | x_t^{1,(i)}) \\
&\text{Compute the normalized weights } \pi_t^{1,(i)} \\
&\text{Compute the weights } w_t^{2,(i)} \propto \pi_{t-1}^{2,(i)} g_2(y_t^{2,(i)} | x_t^{2,(i)}) \\
&\text{Compute the normalized weights } \pi_t^{2,(i)}
\end{aligned}
\tag{13}
$$

Auxiliary Particle Filter

- **idea**: Let y_t also participates in the proposal.
- **how**: In bootstrap sampling, x_t^i is more likely to be generated from $x_{t-1}^{i^j}$ when the value of $\pi_{t-1}^{i^j}$ is high. **Then** , how about let's also give preference to those $x_{t-1}^{i^j}$ where their proposed $x^i \sim x_{t-1}^{i^j}$ can be weighted higher by $p(y_t|x^i)$ as well?
- **in my word**: Have a bit of scouting before sampling!

Auxiliary Particle Filter algorithm

$$\mu_t^i = \mathrm{E}_{x_t}[x_t | x_{t-1}^i], \text{ OR: } \mu_t^i \sim p(x_t | x_{t-1}^i) \tag{14}$$

$$\begin{aligned}
&\text{At time } t\text{, for each particle } i\text{:}\\
&\quad \text{Calculate } \mu_t^i\\
&\quad \text{Compute the weights } w_t^i \propto p(y_t | \mu_t^i)\pi_{t-1}^i\\
&\quad \text{Normalize } w_t^i\\
&\quad \text{Sample } i^j \sim \{w_t^i\}\\
&\quad \text{Sample } x_t^i \sim p(x_t | x_{t-1}^{i^j})\\
&\quad \text{Assign } w_t^i \propto \frac{p(y_t | x_t^i)}{p(y_t | \mu_t^{i_j})}\\
&\quad \text{Normalize } w_t^i \to \pi_t^i
\end{aligned} \tag{15}$$

Why $w_t^i \propto \frac{p(y_t|x_t^i)}{p(y_t|\mu_t^{i^j})}$? The proposal

Looking at the proposal:

$$q(x_t^i, i^j|.) = \underbrace{q(x_t^i|i^j, x_{t-1}, y_{1:t})}_{\text{2: choose } x_t} \underbrace{q(i^j|x_{t-1}, y_{1:t})}_{\text{1: choose the index}} \tag{16}$$

From the algorithm of the previous page:

$$\begin{aligned} &\text{1st Step: choose the index: } q(i^j|x_{t-1}, y_{1:t}) \propto p(y_t|\mu_t^{i^j})\pi_{t-1}^{i^j} \\ &\text{2nd Step: choose the } x_t\text{: } q(x_t^i|i^j, x_{t-1}, y_{1:t}) \equiv p(x_t^i|x_{t-1}^{i^j}) \end{aligned} \tag{17}$$

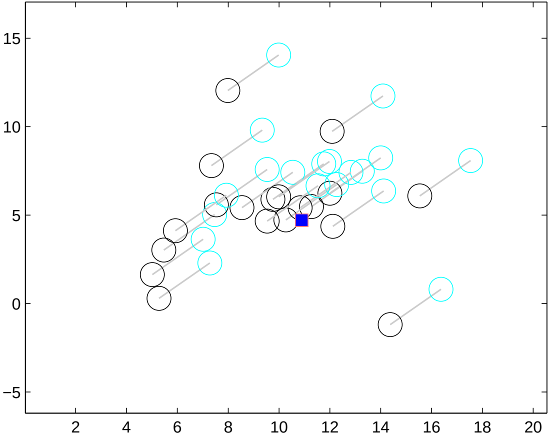
Why $w_t^i \propto \frac{p(y_t|x_t^i)}{p(y_t|\mu_t^{i^j})}$?

Substituting N of the (x^i, i^j) into the following:

$$w_t(x_{1:t}) \propto \pi_{t-1}(x_{1:t-1}) \times \frac{p(y_t|x_t)p(x_t|x_{t-1})}{q(x_t|x_{t-1}, y_t)}$$

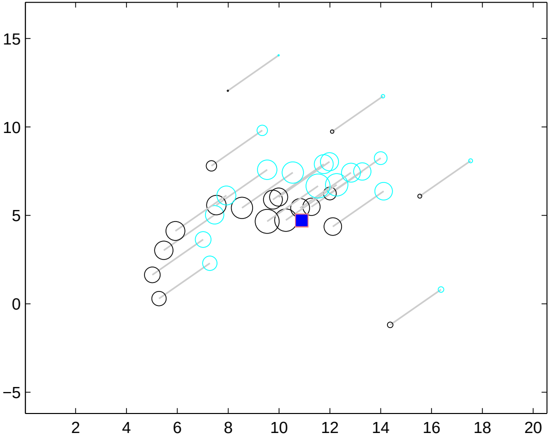
$$\begin{aligned} w_t^i(x_{1:t}) &\propto \pi_{t-1}^{i^j} \times \frac{p(y_t|x_t^i)p(x_t|x_{t-1}^{i^j})}{p(y_t|\mu_t^{i^j})\pi_{t-1}^{i^j}p(x_t^i|x_{t-1}^{i^j})} \\ &= \frac{p(y_t|x_t^i)}{p(y_t|\mu_t^{i^j})} \end{aligned}$$

Representation for $p(x_{t-1}|y_{1:t-1})$ and μ_t^i

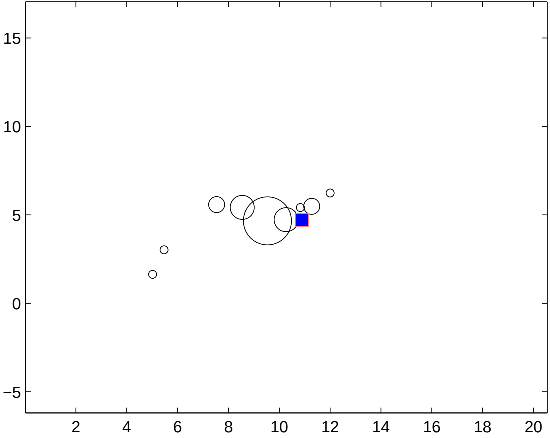


- Light blue circles are μ_t^i for each x_{t-1}^i

New weights: $\propto p(y_t|\mu_t^i)\pi_{t-1}^i$

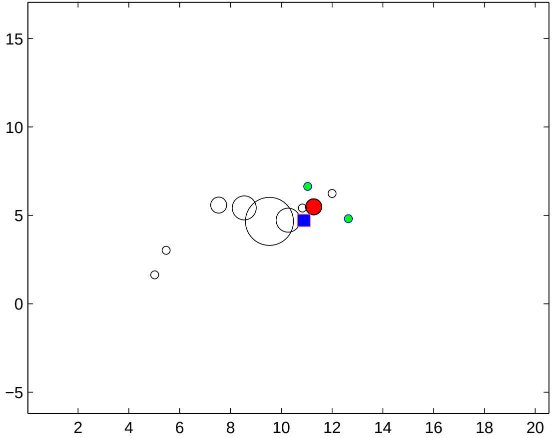


Re-sampling

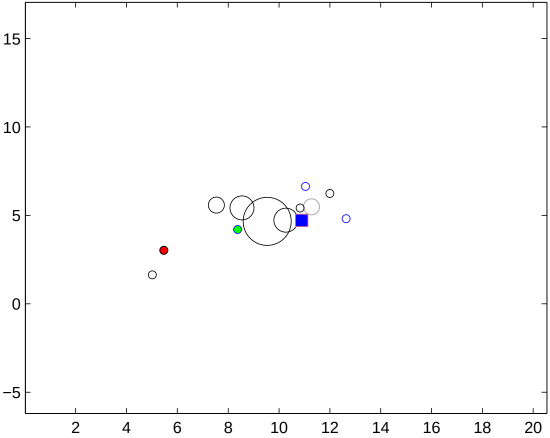


- Size of the circle indicates the number of times $x_{t-1}^{i,j}$ was selected.

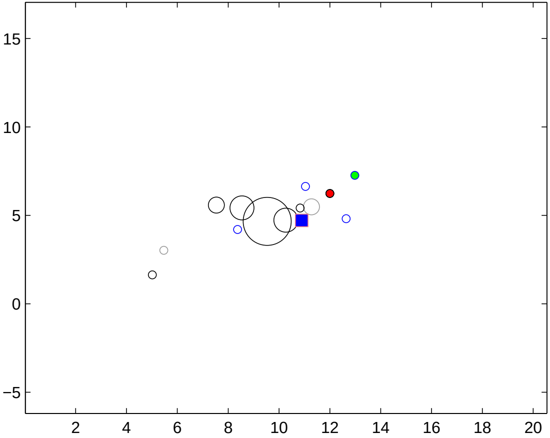
Transition demos



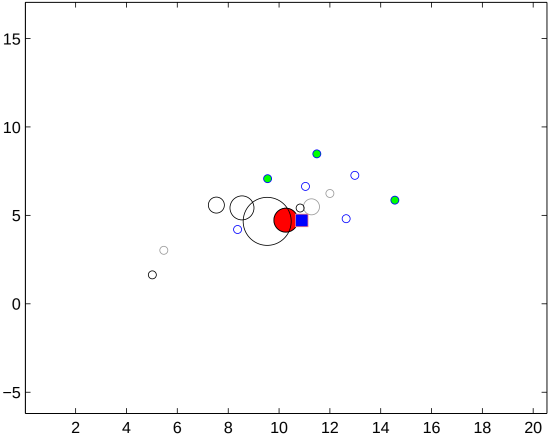
Transition demos



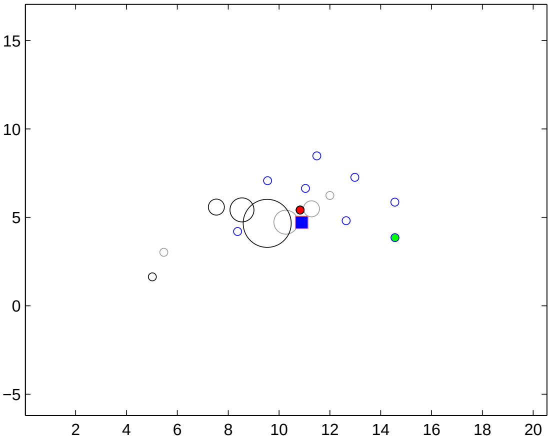
Transition demos



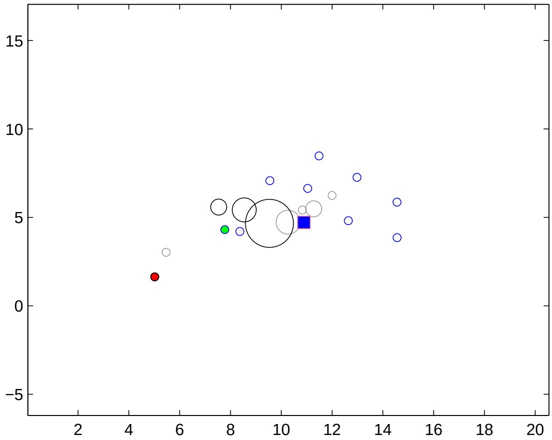
Transition demos



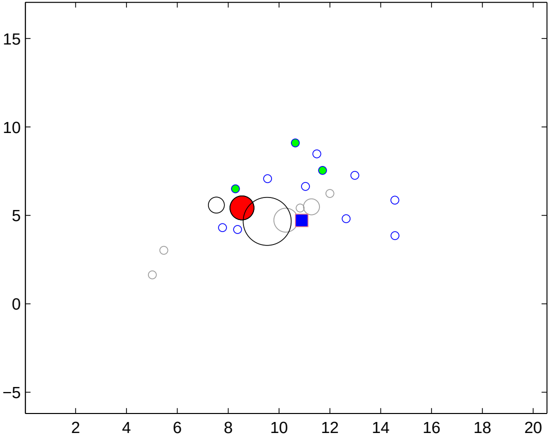
Transition demos



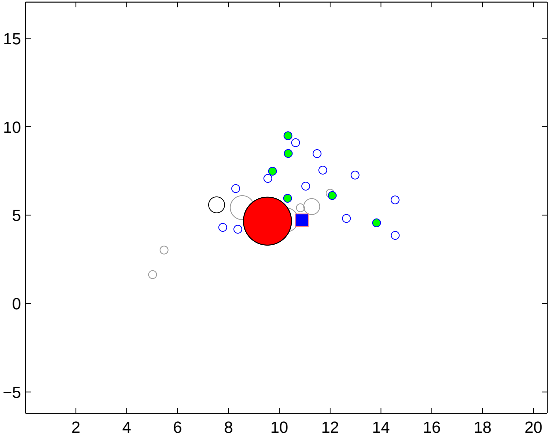
Transition demos



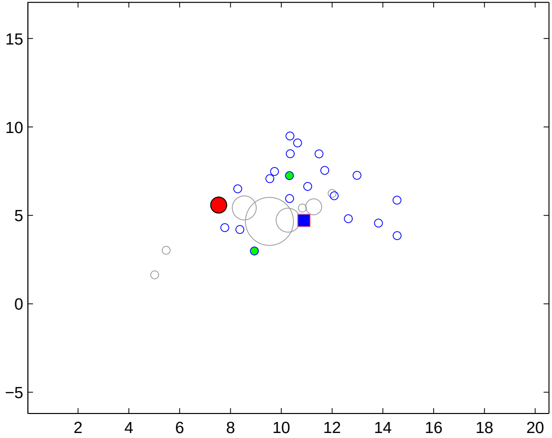
Transition demos



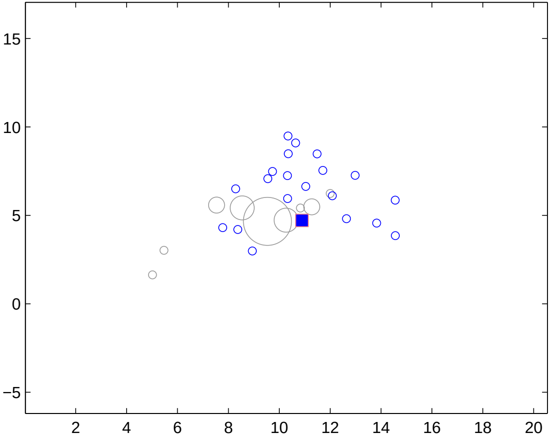
Transition demos



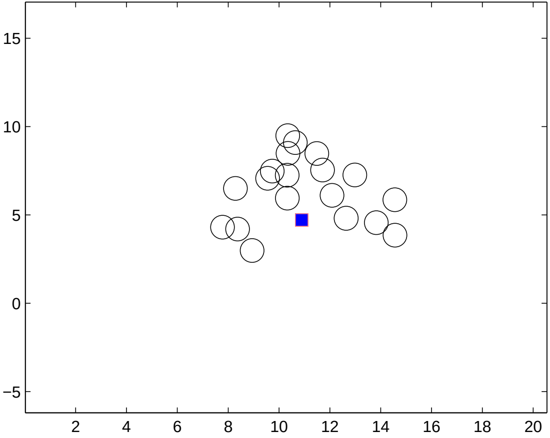
Transition demos



Transition demos

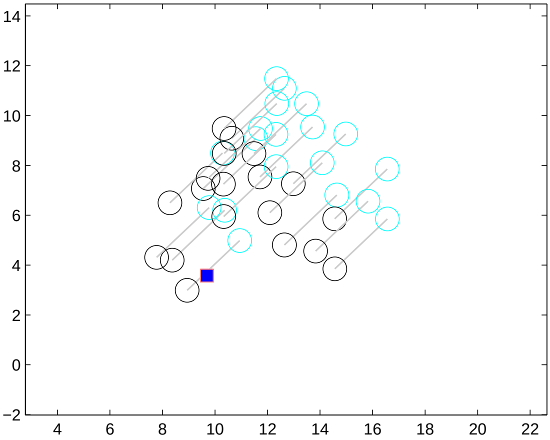


After re-weighting



The above is the representation for $p(x_t|y_{1:t})$ Note that weights are in log scale..

Next t



The above is the representation for $p(x_{t-1}|y_{1:t-1})$ in the next t:

forward-backward recursion (1)

$$
\begin{aligned}
p(x_{1:T}|y_{1:T}) &= p\left(x_T|y_{1:T}\right) \prod_{t=1}^{T-1} p\left(x_t|x_{t+1}, y_{1:T}\right) \\
&= p\left(x_T|y_{1:T}\right) \prod_{t=1}^{T-1} p\left(x_t|x_{t+1}, y_{1:t}\right) \quad x_{t+1} \text{ is markov blanket for } y_{t+1}, \dots \\
&= p\left(x_T|y_{1:T}\right) \prod_{t=1}^{T-1} \frac{p\left(x_t, x_{t+1}|y_{1:t}\right)}{p\left(x_{t+1}|y_{1:t}\right)} \\
&= p\left(x_T|y_{1:T}\right) \prod_{t=1}^{T-1} \frac{f\left(x_{t+1}|x_t\right) p\left(x_t|y_{1:t}\right)}{p\left(x_{t+1}|y_{1:t}\right)}
\end{aligned}
$$

To obtain a specific $p(x_t|y_{1:T})$, we can do:

$$
\begin{aligned}
p(x_t|y_{1:T}) &= \int_{x_{1:t-1},\ x_{t+1:T}} p(x_{1:T}|y_{1:T}) \\
&= \int_{x_{1:t-1},\ x_{t+1:T}} p\left(x_T|y_{1:T}\right) \prod_{t=1}^{T-1} \frac{f\left(x_{t+1}|x_t\right) p\left(x_t|y_{1:t}\right)}{p\left(x_{t+1}|y_{1:t}\right)}
\end{aligned}
$$

foward-backward recursion (2)

however, instead of performing:

$$p(x_t|y_{1:T}) = \int_{x_{1:t-1},\ x_{t+1:T}} p\big(x_T|y_{1:T}\big) \prod_{t=1}^{T-1} \frac{f\big(x_{t+1}|x_t\big)p\big(x_t|y_{1:t}\big)}{p\big(x_{t+1}|y_{1:t}\big)} \quad \text{we perform easilier:}$$

$$\begin{aligned}
p(x_t|y_{1:T}) &= \int_{x_{t+1}} p(x_t, x_{t+1}|y_{1:T})\, \mathrm{d}x_{t+1} \quad \text{standard trick of recursion} \\
&= \int_{x_{t+1}} \frac{p(x_t, x_{t+1}, y_{1:T})}{p(y_{1:T})}\, \mathrm{d}x_{t+1} \\
&= \int_{x_{t+1}} p(x_t|x_{t+1}, y_{1:T}) \frac{p(x_{t+1}, y_{1:T})}{p(y_{1:T})}\, \mathrm{d}x_{t+1} \\
&= \int_{x_{t+1}} p(x_t|x_{t+1}, y_{1:t}) p(x_{t+1}|y_{1:T})\, \mathrm{d}x_{t+1} \\
&= \int_{x_{t+1}} \frac{p(x_t, x_{t+1}, y_{1:t})}{p(x_{t+1}, y_{1:t})} p(x_{t+1}|y_{1:T})\, \mathrm{d}x_{t+1} \\
&= \int_{x_{t+1}} \frac{f(x_{t+1}|x_t) p(x_t|y_{1:t})}{p(x_{t+1}|y_{1:t})} p(x_{t+1}|y_{1:T})\, \mathrm{d}x_{t+1} \\
&= p(x_t|y_{1:t}) \int_{x_{t+1}} \frac{f(x_{t+1}|x_t)}{p(x_{t+1}|y_{1:t})} p(x_{t+1}|y_{1:T})\, \mathrm{d}x_{t+1}
\end{aligned}$$

smoothing algoorithm

$$p(x_t|y_{1:T}) = p(x_t|y_{1:t}) \int_{x_{t+1}} \frac{f(x_{t+1}|x_t)}{p(x_{t+1}|y_{1:t})} p(x_{t+1}|y_{1:T}) \, \mathrm{d}x_{t+1}$$

- given filtering weights $\{(w_t^i, x_t^i)\}_{i=1}^N$:

$$w_{t|T}^i = w_t^i \sum_{k=1}^{N} \frac{f(x_{t+1}^k|x_t^i)}{\sum_{i=1}^N w_t^i f(x_{t+1}^k|x_t^i)} w_{t+1|T}^i$$

- only update weights, no sampling involved:
- does not suffer from degeneracy

generalized two-filter formula

$$
\begin{aligned}
p(x_t|y_{1:T}) &= \frac{p(x_t, y_{1:T})}{p(y_{1:T})} \\
&= \frac{p(x_t, y_{1:T})}{p(y_{1:T})} \quad = \frac{p(y_{t:T}|x_t, y_{1:t-1})p(x_t, y_{1:t-1})}{p(y_{t:T}|y_{1:t-1})p(y_{1:t-1})} \\
&= \frac{p(y_{t:T}|x_t)p(x_t|y_{1:t-1})}{p(y_{t:T}|y_{1:t-1})}
\end{aligned}
$$

References and a set of good place to study sampling

- Christopher Bishop's textbook Pattern Recognition and Machine Learning - include a whole chapter on sampling
- The BUGS project: (http://www.mrc-bsu.cam.ac.uk/bugs/winbugs/contents.shtml)
- For PhD student, Gary Walsh has a great lecture notes on MCMC tutorial, very gentle, called "Markov Chain Monte Carlo and Gibbs Sampling Lecture Notes for EEB 581"
- For SMC stuff, see Doucet and Johansen, "A Tutorial on Particle Filtering and Smoothing: Fifteen years later"
- Arulampalam, M.S. and Maskell, S. and Gordon, N. and Clapp, T, A tutorial on particle filters for online nonlinear/non-Gaussian Bayesian tracking, IEEE Transactions on Signal Processing, 2002
- Pitt, M.K.; Shephard, N. (1999). "Filtering Via Simulation: Auxiliary Particle Filters". Journal of the American Statistical Association (American Statistical Association) 94 (446): 590-591