# Determinantal Point Process and its Time-varying model

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July 23, 2018

# What is DPP?

Start with a **marginal** distribution:

$$\Pr(A \subseteq \mathbf{Y}) = \det(K_A)$$

An example: given $\mathcal{Y} = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$

$$\begin{aligned}\Pr(A \subseteq \mathbf{Y}) \equiv \Pr(A \subseteq \mathbf{Y} \subseteq \mathcal{Y}) &\equiv \Pr\left(\mathbf{Y} \in \{\{1,2,3\},\{1,2,3,4\},\{1,2,3,5\},\{1,2,3,4,5\}\}\right) \\ &= \det(K_A)\end{aligned}$$

$$\begin{aligned}\Pr(A \subseteq \mathbf{Y}) \equiv \Pr(A \subseteq \mathbf{Y} \subseteq \mathcal{Y}) &\equiv \Pr\left(y_1 = 1, y_2 = 1, y_3 = 1\right) \\ &= \sum_{t_4=0}^{1}\sum_{t_5=0}^{1} \Pr\left(y_1 = 1, y_2 = 1, y_3 = 1, y_4 = t_4, y_5 = t_5\right) \\ &= \det(K_A)\end{aligned}$$

# Something about marginal distribution

- $\Pr(A \subseteq \mathbf{Y})$ is marginal, they don't need to add to 1
- it may be possible that, $\Pr(A_1 \subseteq \mathbf{Y}) + \Pr(A_2 \subseteq \mathbf{Y}) > 1$
- $\Pr(\emptyset \subseteq \mathbf{Y}) = \det(K_\emptyset) = 1$ This is obvious, as any $\mathbf{Y}$ is a superset of $\emptyset$.
- $\Pr(i \subseteq \mathbf{Y}) = \det(K_{ii}) = K_{ii}$
- Look at the two element case:

$$\begin{aligned}\Pr(i, j \in \mathbf{Y}) &= \begin{vmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{vmatrix} \\ &= K_{ii}K_{jj} - K_{ij}K_{ji} \\ &= \Pr(i \subseteq \mathbf{Y})\Pr(j \subseteq \mathbf{Y}) - K_{ij}^2\end{aligned}$$

- Off-diagonal elements determine negative correlations between pairs.
- Large values of $K_{ij}$ imply $i$ and $j$ tend **not** co-occur

# Example of $K$ does NOT define DPP

- Any $K, 0 \preceq K \preceq I$ defines a DPP.
- If $K \preceq K'$, that is, $K' - K$ is positive semidefinite.
  Therefore, $\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$ can NOT define DPP, as

$$\left| I - \begin{pmatrix} 1 & -0.5 \\ -0.5 & 1 \end{pmatrix} \right| = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix} \implies \bar{\lambda} = [-0.5, 0.5]^\top$$

- Another way to see the above is incorrect:
  $\mathcal{Y} = \{1, 2\}$

$$\begin{aligned} \Pr(A = \{1\} \subseteq \mathbf{Y}) &\equiv \Pr\left(\mathbf{Y} \in \{\{1\}, \{1,2\}\}\right) \\ &= \det(K_1) = 1 \\ \Pr(A = \{2\} \subseteq \mathbf{Y}) &\equiv \Pr\left(\mathbf{Y} \in \{\{2\}, \{1,2\}\}\right) \\ &= \det(K_2) = 1 \end{aligned}$$

  However,

$$\begin{aligned} \Pr(A = \{1,2\} \subseteq \mathbf{Y}) &\equiv \Pr\left(\mathbf{Y} \in \{\{1,2\}\}\right) \\ &= \det(K_{\{1,2\}}) = 0.75 \end{aligned}$$

  The first two equation says $\{1\}$ and $\{2\}$ must be included; The third equation says both may NOT always be included.

# Example of $K$ define DPP

- Any $K, 0 \preceq K \preceq I$ defines a DPP.
- If $K \preceq K'$, that is, $K' - K$ is positive semidefinite.
  $\begin{bmatrix} 0.3 & -0.1 \\ -0.1 & 0.4 \end{bmatrix}$ **can** define DPP:

$$\left| I - \begin{pmatrix} 0.3 & -0.1 \\ -0.1 & 0.4 \end{pmatrix} \right| = \left| \begin{pmatrix} 0.7 & 0.1 \\ 0.1 & 0.6 \end{pmatrix} \right| \implies \bar{\lambda} = [0.5382, 0.7618]^\top$$

- $\mathcal{Y} = \{1, 2\}$

$$\begin{aligned} \Pr(A = \{1\} \subseteq \mathbf{Y}) &\equiv \Pr\left(Y \in \{\{1\}, \{1,2\}\}\right) \\ &= \det(K_1) = 0.3 \\ \Pr(A = \{2\} \subseteq \mathbf{Y}) &\equiv \Pr\left(Y \in \{\{2\}, \{1,2\}\}\right) \\ &= \det(K_2) = 0.4 \\ \Pr(A = \{1,2\} \subseteq \mathbf{Y}) &\equiv \Pr\left(Y \in \{\{1,2\}\}\right) \\ &= \det(K_{\{1,2\}}) = 0.11 \end{aligned}$$

- So where do rest of probabilities go?

$$\begin{aligned} \Pr(A = \emptyset \subseteq \mathbf{Y}) &\equiv \Pr\left(Y \in \{\emptyset, \{1\}, \{2\}, \{1,2\}\}\right) \\ &= \det(K_\emptyset) = 1 \end{aligned}$$

- Some probabilities mass is assigned to $\emptyset$.

# L-Ensembles

- Marginal distributions does **not** define probabilities in terms of a **particular** set
- i.e., instead of having $\Pr(\mathbf{Y} \subseteq Y)$, we want $\Pr(\mathbf{Y} = Y)$

$$\Pr_L(\mathbf{Y} = Y) \propto \det(L_Y)$$

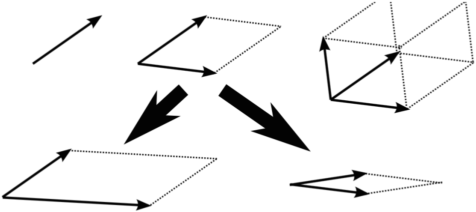
- $L$ must be positive semidefinite.
- Only a statement of proportionality, eigenvalues of $L$ need **not** $< 1$

# Geometry interpretation

$$X = [x_1 \quad x_2 \quad \dots \quad x_n] \implies$$

$$L(x_1, \dots, x_n) = X^\top X = \begin{pmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle & \dots & \langle x_1, x_n \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle & \dots & \langle x_2, x_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \langle x_n, x_2 \rangle & \dots & \langle x_n, x_n \rangle \end{pmatrix}$$

- Gram determinant is the square of the volume of the parallelotope formed by the vectors
- vectors are linearly independent if and only if the Gram determinant is nonzero
- $\Pr_L(Y) \propto \det(L_Y) = \mathrm{Vol}^2\left(\{x_i\}_{i \in Y}\right)$



# Proof for the Geometry interpretation (1)

- in **1-element** case: $\text{Vol}^2(\mathbf{u}_1) = \mathbf{u}_1^\top \mathbf{u}_1$, i.e., length square of a line
- in **k-element** case: $\text{Vol}^2(\mathbf{u}_1 \dots \mathbf{u}_k, \mathbf{u}_{k+1}) = \text{Vol}^2(\mathbf{u}_1, \dots, \mathbf{u}_k)\|\tilde{\mathbf{u}}_{k+1}\|^2$
- $\tilde{\mathbf{u}}_{k+1}$ is the orthogonal projection of $\mathbf{u}_{k+1}$ onto span $(\mathbf{u}_1, \dots, \mathbf{u}_k)$: imagine in in the **2-element** or **3-element** case.
- Let $(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is an $n \times k$ matrix $\mathbf{Y}$:
- Then there exists a vector $\mathbf{c} \in \mathcal{R}^k$ such that:

$$
\begin{aligned}
\mathbf{u}_{k+1} &= \mathbf{u}_{k+1} = \mathbf{Y}\mathbf{c} + \tilde{\mathbf{u}}_{k+1} \\
&= \underbrace{\begin{bmatrix} | & \vdots & | \\ \mathbf{u}_1 & \vdots & \mathbf{u}_k \\ | & \vdots & | \end{bmatrix}}_{\mathbf{Y}} \begin{bmatrix} c_1 \\ \cdot \\ c_k \end{bmatrix} + \tilde{\mathbf{u}}_{k+1} \qquad \text{or } \mathbf{u}_{k+1} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 \dots c_k\mathbf{u}_k + \tilde{\mathbf{u}}_{k+1}
\end{aligned}
$$

# Proof for the Geometry interpretation (2)

- extend $\mathbf{Y}$ to $\mathbf{X}$:

$$\mathbf{X} = [\mathbf{Y} \quad \mathbf{u}_{k+1}] = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \ldots \quad \mathbf{u}_k \quad \mathbf{u}_{k+1}] = [\mathbf{Y} \quad \mathbf{Y}\mathbf{c} + \tilde{\mathbf{u}}_{k+1}]$$

$$\Longrightarrow \mathbf{X}^\top \mathbf{X} = \begin{bmatrix} \mathbf{Y}^\top \mathbf{Y} & \mathbf{Y}^\top(\mathbf{Y}\mathbf{c} + \tilde{\mathbf{u}}_{k+1}) \\ (\mathbf{Y}\mathbf{c} + \tilde{\mathbf{u}}_{k+1})^\top \mathbf{Y} & (\mathbf{Y}\mathbf{c} + \tilde{\mathbf{u}}_{k+1})^\top(\mathbf{Y}\mathbf{c} + \tilde{\mathbf{u}}_{k+1}) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{Y}^\top \mathbf{Y} & \mathbf{Y}^\top \mathbf{Y}\mathbf{c} \\ \mathbf{c}^\top \mathbf{Y}^\top \mathbf{Y} & \mathbf{c}^\top \mathbf{Y}^\top \mathbf{Y}\mathbf{c} + \tilde{\mathbf{u}}_{k+1}^\top \tilde{\mathbf{u}}_{k+1} \end{bmatrix} \qquad \text{since } \mathbf{Y}^\top \tilde{\mathbf{u}}_{k+1} = \mathbf{0}$$

$$= \begin{bmatrix} \mathbf{Y}^\top \mathbf{Y} & \mathbf{Y}^\top \mathbf{Y}\mathbf{c} \\ \mathbf{c}^\top \mathbf{Y}^\top \mathbf{Y} & \mathbf{c}^\top \mathbf{Y}^\top \mathbf{Y}\mathbf{c} + \|\tilde{\mathbf{u}}_{k+1}\|^2 \end{bmatrix}$$

$$= \left[ \begin{bmatrix} \mathbf{Y}^\top \mathbf{Y} \\ \mathbf{c}^\top \mathbf{Y}^\top \mathbf{Y} \end{bmatrix} \quad \left( \begin{bmatrix} \mathbf{Y}^\top \mathbf{Y}\mathbf{c} \\ \mathbf{c}^\top \mathbf{Y}^\top \mathbf{Y}\mathbf{c} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \|\tilde{\mathbf{u}}_{k+1}\|^2 \end{bmatrix} \right) \right]$$

- **Multi-linearity** states:

$$\det([a_1 + b_1, a_2, \ldots, a_k]) = \det\left([a_1, a_2, \ldots, a_k]\right) + \det\left([b_1, a_2, \ldots, a_k]\right)$$

$$\Longrightarrow \left|\mathbf{X}^\top \mathbf{X}\right| = \begin{vmatrix} \mathbf{Y}^\top \mathbf{Y} & \mathbf{Y}^\top \mathbf{Y}\mathbf{c} \\ \mathbf{c}^\top \mathbf{Y}^\top \mathbf{Y} & \mathbf{c}^\top \mathbf{Y}^\top \mathbf{Y}\mathbf{c} \end{vmatrix} + \begin{vmatrix} \mathbf{Y}^\top \mathbf{Y} & \mathbf{0} \\ \mathbf{c}^\top \mathbf{Y}^\top \mathbf{Y} & \|\tilde{\mathbf{u}}_{k+1}\|^2 \end{vmatrix}$$

$$= \mathbf{0} + \begin{vmatrix} \mathbf{Y}^\top \mathbf{Y} & \mathbf{0} \\ \mathbf{c}^\top \mathbf{Y}^\top \mathbf{Y} & \|\tilde{\mathbf{u}}_{k+1}\|^2 \end{vmatrix}$$

$$= \left|\mathbf{Y}^\top \mathbf{Y}\right| \underbrace{\|\tilde{\mathbf{u}}_{k+1}\|^2}_{\mathrm{Vol}^2(\tilde{\mathbf{u}}_{k+1})}$$

# Normalization constant

**Theorem** says,

$$\sum_{A \subseteq Y \subseteq \mathcal{Y}} \det(L_Y) = \det(L + I_{\bar{A}})$$

For example,

$$L = \begin{pmatrix} 2.8599 & -0.4936 & -1.8458 \\ -0.4936 & 2.6264 & -1.1437 \\ -1.8458 & -1.1437 & 2.0522 \end{pmatrix}$$

$$A = \{1, 2\} \implies \bar{A} = \{3\} \implies I_{\bar{A}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, normalisation constant (or partition function) is:

$$\sum_{\emptyset \subseteq Y \subseteq \mathcal{Y}} \det(L_Y) = \sum_{Y \subseteq \mathcal{Y}} \det(L_Y) = \det(L + I_{\bar{\emptyset}}) = \det(L + I)$$

# Conversion to Marginal distribution (1)

$$\Pr_L(\mathbf{Y} = Y) \propto \det(L_Y) \implies \Pr_L(\mathbf{Y} = Y) = \frac{\det(L_Y)}{\det(L_Y + I)}$$

An $L$-ensemble is a DPP, and its marginal kernel is:

$$K = L(L+I)^{-1} = I - (L+I)^{-1}$$

$L(L+I)^{-1} = I - (L+I)^{-1}$ is **true** for any $L$ where $(L+I)^{-1}$ exist, think scalar case:

$$1 - \frac{1}{x+1} = \frac{x+1-1}{x+1} = \frac{x}{x+1}$$

Then,

$$\begin{aligned}
\Pr_L(A \subseteq \mathbf{Y}) &= \frac{\sum_{A \subseteq Y \subseteq \mathcal{Y}} \det(L_Y)}{\sum_{Y \subseteq \mathcal{Y}} \det(L_Y)} \\
&= \frac{\det(L + I_{\bar{A}})}{\det(L + I)} \\
&= \det\left((L + I_{\bar{A}})(L+I)^{-1}\right)
\end{aligned}$$

Since, $\det(A^{-1}) = \frac{1}{\det(A)}$ $\quad$ $\det(AB) = \det(A)\det(B)$

# Conversion to Marginal distribution (2)

$$
\begin{aligned}
\Pr_L(A \subseteq \mathbf{Y}) &= \det\left((L + I_{\bar{A}})(L + I)^{-1}\right) \\
&= \det\left(\underbrace{L(L + I)^{-1}}_{I-(L+I)^{-1}} + I_{\bar{A}}(L + I)^{-1}\right) \\
&= \det\left(I - (L + I)^{-1} + I_{\bar{A}}(L + I)^{-1}\right) \\
&= \det\left(I - (I - I_{\bar{A}})(L + I)^{-1}\right) \\
&= \det\left(I - I_A(L + I)^{-1}\right) \\
&= \det\left(\underbrace{I_A + I_{\bar{A}}}_{I} - I_A(L + I)^{-1}\right) \\
&= \det\left(I_{\bar{A}} + \underline{I_A - I_A(L + I)^{-1}}\right) \\
&= \det\left(I_{\bar{A}} + I_A\left(\underbrace{I - (L + I)^{-1}}_{K}\right)\right)
\end{aligned}
$$

# Conversion to Marginal distribution (3)

$$\Pr_L(A \subseteq \mathbf{Y}) = \det\left(I_{\bar{A}} + I_A\Big(\underbrace{I - (L+I)^{-1}}_{K}\Big)\right)$$

- left multiplication by $I_A$ **zeros out rows** of a matrix except those corresponding to $A$, $\implies$

$$K = \begin{pmatrix} K_{\bar{A}} & K_{\bar{A}A} \\ K_{A\bar{A}} & K_A \end{pmatrix} \implies I_A(K) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{|A|\times|A|} \end{pmatrix} \begin{pmatrix} K_{\bar{A}} & K_{\bar{A}A} \\ K_{A\bar{A}} & K_A \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ K_{A\bar{A}} & K_A \end{pmatrix}$$

- Re-organise:

$$\begin{aligned} \Pr_L(A \subseteq \mathbf{Y}) &= \det(I_{\bar{A}} + I_A K) \\ &= \begin{vmatrix} I_{|\bar{A}|\times|\bar{A}|} & \mathbf{0} \\ K_{A\bar{A}} & K_A \end{vmatrix} \\ &= \det(K_A) \end{aligned}$$

$K = L(L+I)^{-1} = I - (L+I)^{-1}$ is the conversion formula!

# Eigen-value conversion

$$K = L(L+I)^{-1} = I - (L+I)^{-1}$$

- **Properties**

$$\begin{aligned}\lambda_n \in \text{eig}(A) &\implies \lambda_n + 1 \in \text{eig}(A+I)\\ &\implies (\lambda_n)^{-1} \in \text{eig}(A^{-1})\end{aligned}$$

- **Apply** it to $K = I - (L+I)^{-1}$:

$$\begin{aligned}(\lambda_n + 1) \in \text{eig}(L+I) &\implies \frac{1}{\lambda_n + 1} \in \text{eig}\big((L+I)^{-1}\big)\\ &\implies 1 - \frac{1}{\lambda_n + 1} \in \text{eig}\big(I - (L+I)^{-1}\big)\end{aligned}$$

$$1 - \frac{1}{\lambda_n + 1} = \frac{\lambda_n + 1 - 1}{\lambda_n + 1} = \frac{\lambda_n}{\lambda_n + 1}$$

- Therefore,

$$L = \sum_{n=1}^{N} \lambda_n v_n v_n^\top \implies K = \sum_{n=1}^{N} \frac{\lambda_n}{\lambda_n + 1} v_n v_n^\top$$

# Conversions from $K$ to $L$

$$K = L(L+I)^{-1} = I - (L+I)^{-1}$$

$$\begin{aligned}
K = I - (L+I)^{-1} &\implies I - K = (L+I)^{-1} \\
&\implies (L+I)(I-K) = I \\
&\implies L + I - LK - K = I \\
&\implies L(I-K) = K \\
&\implies L = K(I-K)^{-1}
\end{aligned}$$

# Complement

If $\mathbf{Y}$ is distributed as a DPP with marginal kernel K, then $\mathcal{Y} - \mathbf{Y}$ is also distributed as a DPP, with marginal kernel $\bar{K} = I - K$,

$$\Pr(A \cap \mathbf{Y} = \emptyset) = \det(\bar{K}_A) = \det(I - K_A)$$

For example:

$$K = \begin{pmatrix} 0.4 & 0.1 & -0.1 \\ 0.05 & 0.5 & 0.1 \\ -0.01 & 0.1 & 0.3 \end{pmatrix}, A = \{1,2\}, \bar{A} = \{3\}$$

$$\bar{K} = I - K = \begin{pmatrix} 0.6 & -0.1 & 0.1 \\ -0.05 & 0.5 & -0.1 \\ 0.01 & -0.1 & 0.7 \end{pmatrix} \implies \bar{K}_{A=\{1,2\}} = \begin{pmatrix} 0.6 & -0.1 \\ -0.05 & 0.5 \end{pmatrix}$$

It's easy to see that $\bar{K}_A = (I - K_A)$

# Complement

$$
\begin{aligned}
\Pr(i, j \notin \mathbf{Y}) &= 1 - \Pr(i, j \in \mathbf{Y}) \\
&= 1 - \big( \Pr(i \in \mathbf{Y}) + \Pr(j \in \mathbf{Y}) - \Pr(i, j \in \mathbf{Y}) \big) \\
&= 1 - \Pr(i \in \mathbf{Y}) - \Pr(j \in \mathbf{Y}) + \Pr(i, j \in \mathbf{Y}) \\
&\leq 1 - \Pr(i \in \mathbf{Y}) - \Pr(j \in \mathbf{Y}) + \Pr(i \in \mathbf{Y}) \Pr(j \in \mathbf{Y}) \\
&= 1 - \Pr(i \in \mathbf{Y}) + (1 - \Pr(j \in \mathbf{Y})) - 1 + (1 - \Pr(i \notin \mathbf{Y}))(1 - \Pr(j \notin \mathbf{Y})) \\
&= \Pr(i \notin \mathbf{Y}) + \Pr(j \notin \mathbf{Y}) - 1 + \underline{(1 - \Pr(i \notin \mathbf{Y}))(1 - \Pr(j \notin \mathbf{Y}))} \\
&= \Pr(i \notin \mathbf{Y}) + \Pr(j \notin \mathbf{Y}) - 1 + \underline{1 - \Pr(i \notin \mathbf{Y}) - \Pr(j \notin \mathbf{Y}) + \Pr(i \notin \mathbf{Y}) \Pr(j \notin \mathbf{Y})} \\
&= \Pr(i \notin \mathbf{Y}) \Pr(j \notin \mathbf{Y})
\end{aligned}
$$

Complement of a diversifying process also encourage diversity.

# Larger marginal distribution

$$K \preceq K' \implies \det\left(K_A\right) \leq \det\left(K'_A\right) \qquad \forall A \subseteq \mathcal{Y}$$

- DPP defined by $K'$ is larger than the one defined by $K$
- in the sense that it assigns higher marginal probabilities to every set $A$.

# Quality vs Diversity

- Think of a Gram matrix, let each column matrix $x_i$:

$$q_i = \|x_i\|_{L_2} \qquad \phi_i = \frac{x_i}{q_i} \implies \|\phi_i\| = 1$$

- Let $Q = \begin{bmatrix} q_i & 0 & \dots & 0 \\ 0 & q_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & q_n \end{bmatrix} \implies [q_1\phi_1 \quad q_2\phi_2 \quad \dots \quad q_n\phi_n] = \Phi Q$

$$\begin{aligned} L(x_1, \dots, x_n) &= X^\top X = (\Phi Q)^\top (\Phi Q) = Q^\top \Phi^\top \Phi Q \\ \implies L_{ij} &= q_i \phi_i^\top \phi_j q_j \end{aligned}$$

- $S_{ij} \equiv \phi_i^\top \phi_j \in [-1, 1] \implies S_{ij} = \frac{L_{ij}}{\sqrt{L_{ii} L_{jj}}}$
- $\Pr_L(\mathbf{Y} = Y)$ can be viewed as the product of four determinants

$$\Pr_L(\mathbf{Y} = Y) \propto \left( \prod_{i \in Y} q_i^2 \right) \det(S_Y)$$

# Conditional (1)

- for $(B \subseteq \mathcal{Y}) \cap A = \emptyset$:

$$
\begin{aligned}
\Pr_L(\mathbf{Y} = B | A \cap \mathbf{Y} = \emptyset) &= \frac{\Pr_L\left((\mathbf{Y} = B) \cap (A \cap \mathbf{Y} = \emptyset)\right)}{\Pr_L(A \cap \mathbf{Y} = \emptyset)} \\
&= \frac{\underbrace{\Pr_L\left(A \cap \mathbf{Y} = \emptyset | \mathbf{Y} = B\right)}_{\Pr = 1 \text{ when } B \cap A = \emptyset} \Pr_L\left(\mathbf{Y} = B\right)}{\Pr_L(A \cap \mathbf{Y} = \emptyset)} \\
&= \frac{\Pr_L\left(\mathbf{Y} = B\right)}{\Pr_L(A \cap \mathbf{Y} = \emptyset)}
\end{aligned}
$$

- all combination of $\mathbf{Y} \subseteq \bar{A}$
- **note** the usual form is defined for $\Pr_L(\bar{A} \subseteq \mathbf{Y}) = \det(L + I_{\bar{A}})$
- normally, $\Pr_L$ is defined in terms of equality sign. However, it is now defined in terms of a set. Therefore, we need the summation of all the equal terms:

$$
\begin{aligned}
&= \frac{\frac{\det(L_B)}{\det\left(L_{\mathcal{Y}} + I\right)}}{\frac{\sum_{B' : B' \cap A = \phi} \det(L_{B'})}{\det\left(L_{\mathcal{Y}} + I\right)}} = \frac{\det(L_B)}{\underbrace{\sum_{B' : B' \cap A = \phi} \det(L_{B'})}_{\text{this is the normalisation constant from } \mathcal{Y} \to \bar{A}}} \\
&= \frac{\det(L_B)}{\det(L_{\bar{A}} + I)}
\end{aligned}
$$

# Conditional (2)

For $(B \subseteq \mathcal{Y}) \cap A = \emptyset$:

$$
\begin{aligned}
\Pr_L(\mathbf{Y} = A \cup B | A \subseteq \mathbf{Y}) &= \frac{\Pr_L\left((\mathbf{Y} = A \cup B) \cap (A \subseteq \mathbf{Y})\right)}{\Pr_L(A \subseteq \mathbf{Y})} \\
&= \frac{\underbrace{\Pr_L\left(A \subseteq \mathbf{Y} | \mathbf{Y} = A \cup B\right)}_{\Pr=1} \Pr_L\left(\mathbf{Y} = A \cup B\right)}{\Pr_L(A \subseteq \mathbf{Y})} \\
&= \frac{\Pr_L\left(\mathbf{Y} = A \cup B\right)}{\Pr_L(A \subseteq \mathbf{Y})} \\
&= \frac{\det(L_{A \cup B})}{\det(L + I_{\bar{A}})}
\end{aligned}
$$

# Sampling DPP:

- Can you sample directly from $\mathcal{Y} \sim L$? Computationally impossible, as there are $2^N$ combinations.
- You can NOT sample from $K$ either. Since $K$ is defined as marginal, you can NOT add up all cases.
- The solution, express DPP in terms of mixture of **elementary** DPPs:

$$
\begin{aligned}
&\frac{1}{\det(L+I)} \sum_{J \subseteq \{1,2,\dots,N\}} \mathcal{P}^{V_J} \prod_{n \in J} \lambda_n \\
=&\frac{1}{\det(L+I)} \sum_{J \subseteq \{1,2,\dots,N\}} \det\left(K^{V_J}\right) \prod_{n \in J} \lambda_n
\end{aligned}
$$

- of course, we assume sampling $\det\left(K^{V_J}\right)$ is a lot faster in computation

# Sampling DPP:

- Mixing weights of using $V_J$ to construct **elementary** DPP is: $\frac{\prod_{n \in J} \lambda_n}{\det(L+I)} = \frac{\prod_{n \in J} \lambda_n}{\prod_{n=1}^{N}(\lambda_n+1)}$
- For example, let $J = \{1, 3, 5\}$, it's mixing weights are $\frac{\lambda_1 \lambda_3 \lambda_5}{\prod_{n=1}^{N}(\lambda_n+1)}$
- Now, we can decide the probability of inclusion of a single $V_j$ to construct **elementary** DPP! For example, let $N = 3$, and we need to decide the inclusion of the element 1:

$$
\begin{aligned}
&\frac{\lambda_1 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_2\lambda_3}{(\lambda_1+1)(\lambda_2+1)(\lambda_3+1)} = \frac{\lambda_1(1 + \lambda_2 + \lambda_3 + \lambda_2\lambda_3)}{(\lambda_1+1)(\lambda_2+1)(\lambda_3+1)} \\
=&\frac{\lambda_1(1+\lambda_2)(1+\lambda_3)}{(\lambda_1+1)(\lambda_2+1)(\lambda_3+1)} = \frac{\lambda_1}{(\lambda_1+1)}
\end{aligned}
$$

# Elementary DPP:

- A DPP, is called **elementary** if every eigenvalue of its marginal kernel is $\in \{0, 1\}$
- We write $\mathcal{P}^V = \det\left(K^V\right)$, where $V$ is a set of **orthonormal** vectors, to denote an elementary DPP with marginal kernel $K^V = \sum_{v \in V} vv^T$
- **example**: $V \in \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$K^V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

- **example**: $V \in \left\{ \begin{bmatrix} -0.5735 \\ 0.7781 \\ -0.2562 \end{bmatrix}, \begin{bmatrix} -0.3243 \\ 0.0716 \\ 0.9432 \end{bmatrix}, \begin{bmatrix} 0.7523 \\ 0.6240 \\ 0.2113 \end{bmatrix} \right\}$

$$\begin{aligned} K^V = \begin{bmatrix} 0.3945 & -0.0557 & -0.4856 \\ -0.0557 & 0.9949 & -0.0447 \\ -0.4856 & -0.0447 & 0.6106 \end{bmatrix} &= 1 \times \begin{bmatrix} -0.5735 \\ 0.7781 \\ -0.2562 \end{bmatrix} \begin{bmatrix} -0.5735 & 0.7781 & -0.2562 \end{bmatrix} \\ &+ 0 \times \begin{bmatrix} -0.3243 \\ 0.0716 \\ 0.9432 \end{bmatrix} \begin{bmatrix} -0.3243 & 0.0716 & 0.9432 \end{bmatrix} \\ &+ 1 \times \begin{bmatrix} 0.7523 \\ 0.6240 \\ 0.2113 \end{bmatrix} \begin{bmatrix} 0.7523 & 0.6240 & 0.2113 \end{bmatrix} \end{aligned}$$

- $K^V$ is a sum of a set of rank one matrix, each constructed from an ortho-normal set
- $K^V$ is still a valid KPP marginal kernel

# Multi-Linearity

- We let $W_J = \sum_{n \in J} W_n$, where each $W_n$ is rank-one matrix:

$$\det(W_J) = \det\left(\sum_{n \in J} W_n\right) = \det\left([(W_J)_1, (W_J)_2, \ldots, (W_J)_k]\right)$$

$$= \det\left(\left[\left(\sum_{n \in J} W_n\right)_1, (W_J)_2, \ldots, (W_J)_k\right]\right) \qquad \text{expand first term}$$

for example:

$$W_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 6 & 4 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$W_J = W_1 + W_2 = \begin{bmatrix} 10 & 8 \\ 8 & 8 \end{bmatrix}$$

$$\left(\sum_{n \in J} W_n\right)_1 = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$$

# Multi-Linearity

- because **Multi-linearity** states:

$$\det([a_1 + b_1, a_2, \ldots, a_k]) = \det\left([a_1, a_2, \ldots, a_k]\right) + \det\left([b_1, a_2, \ldots, a_k]\right)$$

- Therefore,

$$\begin{aligned}\det(W_J) &= \det\left(\left[\left(\sum_{n \in J} W_n\right)_1, (W_J)_2, \ldots, (W_J)_k\right]\right) \\ &= \sum_{n \in J} \det\left([(W_n)_1, (W_J)_2, \ldots, (W_J)_k]\right)\end{aligned}$$

- Now, we repeat the same thing for the second term and all subseqnet times,
- But we can't use $n$ for different columns, we have to give a different index $n_i \in J \quad \forall i$:

$$\det(W_J) = \sum_{n_1 \in J} \sum_{n_2 \in J} \cdots \sum_{n_k \in J} \det\left(\left[(W_{n_1})_1, (W_{n_2})_2, \ldots, (W_{n_k})_k\right]\right)$$

- can be visualized as:
  - form a third order tensor, where each layer correspond to $W_n$
  - pick one column from each of the layer to form a matrix
  - compute its determinant
  - sum the determinants of all combinations

# Elementary DPP:

$$\det(W_J) = \sum_{n_1 \in J} \sum_{n_2 \in J} \cdots \sum_{n_k \in J} \det\left(\left[(W_{n_1})_1, (W_{n_2})_2, \ldots, (W_{n_k})_k\right]\right)$$

- not every term is non-zero.
- Since $W_n$ is rank one matrix, $(W_n)_i$ and $(W_n)_j$ are linearly dependant. Therefore, the determinant of any matrix containing two or more columns of the **same** $W_n$ is zero, for example,

$$\det(W_J) = \det\left(\left[(W_{n_1})_1, (W_{n_1})_2, \ldots, (W_{n_k})_k\right]\right) = 0$$

- Thus the terms in the sum vanish unless $n_1, n_2, \ldots n_k$ are distinct.

$$\begin{aligned}
\det(W_J) &= \sum_{n_1 \in J} \sum_{n_2 \in J} \cdots \sum_{n_k \in J} \det\left(\left[(W_{n_1})_1, (W_{n_2})_2, \ldots, (W_{n_k})_k\right]\right) \\
&= \underbrace{\sum_{n_1 \in J} \sum_{n_2 \in J} \cdots \sum_{n_k \in J}}_{n_1, n_2, \ldots, n_k \text{ are distinct}} \det\left(\left[(W_{n_1})_1, (W_{n_2})_2, \ldots, (W_{n_k})_k\right]\right) \\
&= \sum_{\underbrace{n_1, n_2, \ldots, n_k \in J}_{\text{are distinct}}} \det\left(\left[(W_{n_1})_1, (W_{n_2})_2, \ldots, (W_{n_k})_k\right]\right)
\end{aligned}$$

# Mixture of elementary DPPs

- We need to show a DPP with kernel $L = \sum_{n=1}^{N} \lambda_n v_n v_n^\top$ is a mixture of elementary DPPs:

$$\frac{1}{\det(L+I)} \sum_{J \subseteq 1,2,\ldots,N} \mathcal{P}^{V_J} \prod_{n \in J} \lambda_n$$

- Start from from mixture of elementary DPPs:

$$\begin{aligned}
&\frac{1}{\det(L+I)} \sum_{J \subseteq \{1,2,\ldots,N\}} \mathcal{P}^{V_J} \prod_{n \in J} \lambda_n \\
&= \frac{1}{\det(L+I)} \sum_{J \subseteq \{1,2,\ldots,N\}} \det\left(K^{V_J}\right) \prod_{n \in J} \lambda_n \\
&= \frac{1}{\det(L+I)} \sum_{J \subseteq \{1,2,\ldots,N\}} \det\left(\sum_{n \in J} W_n\right) \prod_{n \in J} \lambda_n \\
&= \frac{1}{\det(L+I)} \sum_{J \subseteq \{1,2,\ldots,N\}} \sum_{\underbrace{n_1, n_2, \ldots, n_k \in J}_{\text{are distinct}}} \det\left(\left[(W_{n_1})_1, (W_{n_2})_2, \ldots, (W_{n_k})_k\right]\right) \prod_{n \in J} \lambda_n
\end{aligned}$$

$$\frac{1}{\det(L+I)} = \frac{1}{\det(L+I)} \sum_{J \subseteq \{1,2,\ldots,N\}} \underbrace{\sum_{n_1,n_2,\ldots,n_k \in J}}_{\text{are distinct}} \det\left(\left[(W_{n_1})_1, (W_{n_2})_2, \ldots, (W_{n_k})_k\right]\right) \prod_{n \in J} \lambda_n$$

- For the outter loop, $\sum_{J \subseteq \{1,2,\ldots,N\}}$ when $|J| < k$, then, the inner loop is zero.
- Therefore, we need $J \subseteq \{1, 2, \ldots, N\} \to J \supseteq \{n_1, n_2, \ldots n_k\}$
- We can also remove $\in J$ from $n_1, n_2, \ldots n_k \in J$
- By swapping the inner and outter loops, we have:

$$\begin{aligned}
\frac{1}{\det(L+I)} &= \frac{1}{\det(L+I)} \sum_{J \subseteq \{1,2,\ldots,N\}} \underbrace{\sum_{n_1,n_2,\ldots,n_k \in J}}_{\text{are distinct}} \det\left(\left[(W_{n_1})_1, (W_{n_2})_2, \ldots, (W_{n_k})_k\right]\right) \prod_{n \in J} \lambda_n \\
&= \frac{1}{\det(L+I)} \underbrace{\sum_{n_1,n_2,\ldots,n_k}}_{\text{are distinct}} \det\left(\left[(W_{n_1})_1, (W_{n_2})_2, \ldots, (W_{n_k})_k\right]\right) \sum_{J \supseteq \{n_1,n_2,\ldots,n_k\}} \prod_{n \in J} \lambda_n
\end{aligned}$$

# Mixture of elementary DPPs

$$\Pr_L = \frac{1}{\det(L+I)} \underbrace{\sum_{n_1, n_2, \ldots, n_k}}_{\text{are distinct}} \det\left([(W_{n_1})_1, (W_{n_2})_2, \ldots, (W_{n_k})_k]\right) \sum_{J \supseteq \{n_1, n_2, \ldots, n_k\}} \prod_{n \in J} \lambda_n$$

- For example, let $J \subseteq \{1,2,3,4,5\}$, and let $\{n_1, n_2, \ldots n_k\} = \{1,2,3\}$
- Then, $T = J \supseteq \{n_1, n_2, \ldots, n_k\} = \{\{1,2,3\}, \{1,2,3,4\}, \{1,2,3,5\}, \{1,2,3,4,5\}\}$:

$$\begin{aligned}
\sum_{J \supseteq \{n_1, n_2, \ldots, n_k\}} \prod_{n \in J} \lambda_n &= \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_3\lambda_5 + \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 \qquad \text{using the example} \\
&= \lambda_1\lambda_2\lambda_3(1 + \lambda_4 + \lambda_5 + \lambda_4\lambda_5) \\
&= \lambda_1\lambda_2\lambda_3(1+\lambda_4)(1+\lambda_5) \qquad \text{this step is the key} \\
&= \lambda_1\lambda_2\lambda_3(1+\lambda_4)(1+\lambda_5)\frac{(\lambda_1+1)(\lambda_2+1)(\lambda_3+1)(\lambda_4+1)(\lambda_5+1)}{(\lambda_1+1)(\lambda_2+1)(\lambda_3+1)(\lambda_4+1)(\lambda_5+1)} \\
&= \frac{\lambda_1}{\lambda_1+1}\frac{\lambda_2}{\lambda_2+1}\frac{\lambda_3}{\lambda_3+1}(\lambda_1+1)(\lambda_2+1)(\lambda_3+1)(\lambda_4+1)(\lambda_5+1) \\
&= \frac{\lambda_{n_1}}{\lambda_{n_1}+1} \cdots \frac{\lambda_{n_k}}{\lambda_{n_k}+1} \prod_{n=1}^{N}(\lambda_n+1) \qquad \text{if we generalise}
\end{aligned}$$

# Mixture of elementary DPPs

$$\Pr_L = \frac{1}{\det(L+I)} \underbrace{\sum_{n_1,n_2,\ldots,n_k}}_{\text{are distinct}} \det\left(\left[(W_{n_1})_1,(W_{n_2})_2,\ldots,(W_{n_k})_k\right]\right) \sum_{J \supseteq \{n_1,n_2,\ldots,n_k\}} \prod_{n\in J} \lambda_n$$

$$= \frac{1}{\prod_{n=1}^N(\lambda_n+1)} \underbrace{\sum_{n_1,n_2,\ldots,n_k}}_{\text{are distinct}} \det\left(\left[(W_{n_1})_1,(W_{n_2})_2,\ldots,(W_{n_k})_k\right]\right) \frac{\lambda_{n_1}}{\lambda_{n_1}+1}\cdots\frac{\lambda_{n_k}}{\lambda_{n_k}+1}\prod_{n=1}^N(\lambda_n+1)$$

$$= \underbrace{\sum_{n_1,n_2,\ldots,n_k}}_{\text{are distinct}} \det\left(\left[(W_{n_1})_1,(W_{n_2})_2,\ldots,(W_{n_k})_k\right]\right) \frac{\lambda_{n_1}}{\lambda_{n_1}+1}\cdots\frac{\lambda_{n_k}}{\lambda_{n_k}+1}$$

$$= \underbrace{\sum_{n_1,n_2,\ldots,n_k}}_{\text{are distinct}} \det\left(\left[(W_{n_1})_1\frac{\lambda_{n_1}}{\lambda_{n_1}+1},(W_{n_2})_2\frac{\lambda_{n_2}}{\lambda_{n_2}+1},\ldots,(W_{n_k})_k\frac{\lambda_{n_k}}{\lambda_{n_k}+1}\right]\right)$$

$$= \det\left(\sum_{n=1}^N \frac{\lambda_n}{\lambda_n+1}W_n\right) = \det(K_A)$$

# Sequential DPP

- Data can be stream in, for example, news sourced from various places, but they do **not** change significantly between consecutive times.
- Our aim is to select the most diverse subset of news to display at each time interval - to avoid show similar pieces. This is done through DPP
- Rather than applying separate DPP sampling at each time stamp, we use Sequential Monte Carlo.
- We have a set of $\{\{y_k^{(i)}, W_k^{(i)}\}_{i=1}^N\}_{k=1}^t$ representing a distribution of selected news articles.

# Vanilla Importance Sampling

- We do **not** know how to sample $\gamma_n$:

$$\pi_n(x) = \frac{\gamma_n(x)}{Z_n}$$

- but instead, we sample $N$ particles $\{X_n^{(i)}\}$ from $\eta_n$, and have:

$$\eta_n^N(\mathrm{d}x) = \frac{1}{N}\sum_{i=1}^{N}\delta_{X_n^{(i)}}(\mathrm{d}x)$$

- and compute their weights:

$$w_n(x) = \frac{\gamma_n(x)}{\eta_n(x)}$$

- These "weighted" particles can be used to approximate:
  - Expectation of a function $\Psi(x)$ : $\mathbb{E}_{\pi_n}(\Psi) = Z_n^{-1}\int \Psi(x)w_n(x)\eta_n(x)\mathrm{d}x$
  - Partition function: $Z_n = \int w_n(x)\eta_n(x)\mathrm{d}x$

# Vanilla Sequential Importance Sampling

- At time $n-1$, we have $N$ particles $\{X_{n-1}^{(i)}\}$ distributed according to $\eta_{n-1}$
- We propose to move these particles using a Markov kernel $K_n$ with associated density $K_n(x, x')$
- Particles are are marginally distributed according to:

$$\eta_n(x') = \int_E \eta_n(x) K_n(x, x') \mathrm{d}x$$

- If $\eta_n$ can be computed point-wise, then it is possible to use the standard IS estimates of $\pi_n$ and $Z_n$
- The **limitation** is that, in most cases, it's impossible to compute importance distribution $\eta_n(x_n)$:

$$\eta_n(x_n) = \int \eta_1 \prod_{k=1}^{n} K_k(x_{k-1}, x_k) \mathrm{d}x_{1:n-1}$$

- Sequential Monte Carlo (SMC) performs Importance Sampling between:
  - artificial joint sequential distribution

$$\widetilde{\pi}_t(X_{1:t}) = \frac{\widetilde{\gamma}(X_{1:t})}{Z_t} = \frac{\gamma_t(x_t)\prod_{k=1}^{t-1} L_k(x_{k+1}, x_k)}{Z_t}$$

  - **and** joint importance distribution:

$$\eta_t(X_{1:t}) = \eta_1(x_1)\prod_{k=2}^{t} K_k(x_{k-1}, x_k)$$

- $\widetilde{\pi}_n(x_{1:n})$ admits $\pi_n(x_n)$ as a marginal by construction.
- $L_k(x_{k+1}, x_k)$ are backward Markov kernels.
- $K_k(x_{k-1}, x_k)$ are forward Markov kernels.

- The unnormalized importance weights:

$$w_t(X_{1:t}) = \frac{\widetilde{\gamma}_t(X_{1:t})}{\eta_t(X_{1:t})} = w_{t-1}(x_{1:t-1})\widetilde{w}_t(x_{t-1}, x_t)$$

- (unnormalized) incremental weight:

$$\widetilde{w}_t(x_{t-1}, x_t) = \frac{\gamma_t(x_t)L_{t-1}(x_t, x_{t-1})}{\gamma_{t-1}(x_{t-1})K_t(x_{t-1}, x_t)}$$

- $\pi_t(x_t)$ is **marginal distribution** of $\widetilde{\pi}_t(X_{1:t})$, it can be approximated by sampled $N$ particle-weight pairs $\{X_{1:t}^{(i)}, W_{1:t}^{(i)}\}_{i=1}^N$,

$$\pi_t^N = \sum_{i=1}^N W_t^{(i)}\delta(X_t^{(i)})$$

# Sub-optimal backward kernel

- When $\pi_t \approx \pi_{t-1}$ a good approximation for optimal $L_{t-1}$:

$$L_{t-1}(x_t, x_{t-1}) = \frac{\pi_t(x_{t-1})K_t(x_{t-1}, x_t)}{\pi_t(x_t)}$$

- Substituting it into $\widetilde{w}_t(x_{t-1}, x_t)$, the unnormalized incremental weight is reformulated as:

$$\begin{aligned}
\widetilde{w}_t(x_{t-1}, x_t) &= \frac{\gamma_t(x_t)L_{t-1}(x_t, x_{t-1})}{\gamma_{t-1}(x_{t-1})K_t(x_{t-1}, x_t)} \\
&= \frac{\gamma_t(x_t)}{\gamma_{t-1}(x_{t-1})K_t(x_{t-1}, x_t)} \times \frac{\pi_t(x_{t-1})K_t(x_{t-1}, x_t)}{\pi_t(x_t)} \\
&= \frac{\gamma_t(x_{t-1})}{\gamma_{t-1}(x_{t-1})}
\end{aligned}$$

- Require the following to make it work:

$$\begin{aligned}
&\gamma_{t-1} \text{ or } \pi_{t-1} \\
&\gamma_t \text{ or } \pi_t \\
&\{x_{t-1}^{(i)}\}_{i=1}^{N} \text{ at time stamp } t-1
\end{aligned}$$

# Bring DPP into SMC framework: Sequential DPP

$$\pi_t(\mathbf{Y} = y_t) = \frac{\det(M_{t,y_t})}{\det(M_t + I)}$$

- The matrix $M_t$ is constructed from data $X_t = (x_{t1}, \ldots, x_{tN_t})^T$ with each $x_{ti} \in R^D$ using $M = X_t X_t^T$
- assume at $t$, dataset $X_t$ differs from $X_{t+1}$ by a only a few elements $\implies \pi_{t-1} \approx \pi_t$
- For simplicity, we assume that $|X_t| = |X_{t-1}|$.
- at time $t = 1$, we samples $\left\{\mathbf{Y}_1^{(i)} \sim \det\left(L_{1,y_1^{(i)}}\right)\right\}_{i=1}^N$ using a **fast DPP sampling** (Kang 2013)
- at each time $t > 1$, we update these samples from $\{y_{t-1}^{(i)}\}_{i=1}^N$ using SMC scheme.

# Incremental weight for sequential DPPs

$$\widetilde{w}_t(y_{t-1}, y_t) = \frac{\det(M_{t,y_{t-1}})/\det(M_t + I)}{\det(M_{t-1,y_{t-1}})/\det(M_{t-1} + I)}$$
$$\propto \det(M_{t,y_{t-1}})/\det(M_{t-1,y_{t-1}})$$

- difference between $\det(M_{t,x_{t-1}})$ and $\det(M_{t-1,y_{t-1}})$ is small
- Let $\mathbb{M}^{\mathbf{c},\mathbf{c}}$ denote the shared sub-matrix between $M_{t,y_{t-1}}$ and $M_{t-1,y_{t-1}}$
- $M_{t,y_{t-1}} = \begin{bmatrix} \mathbb{M}^{\mathbf{c},\mathbf{c}} & M^{c,t} \\ M^{t,c} & M^{t,t} \end{bmatrix}$
- $M_{t-1,y_{t-1}} = \begin{bmatrix} \mathbb{M}^{\mathbf{c},\mathbf{c}} & M^{c,t-1} \\ M^{t-1,c} & M^{t-1,t-1} \end{bmatrix}$
- The **trick** is determinant ratio can be computed **efficiently** by applying determinant formula of partitioned block matrices.
- **no need** to compute nominator nor denominator explicitly.
- efficiently compute incremental weights:

$$\widetilde{w}_t(y_{t-1}, y_t) \propto \frac{\det(M^{c,t} - M^{tc}(\mathbb{M}^{\mathbf{c},\mathbf{c}})^{-1}M^{c,t})}{\det(M^{c,t-1} - M^{t-1,c}(\mathbb{M}^{\mathbf{c},\mathbf{c}})^{-1}M^{c,t-1})}$$

# Computation at the Inverse

$$\widetilde{w}_t(y_{t-1}, y_t) \propto \frac{\det(M^{c,t} - M^{tc}(\mathbb{M}^{\mathbf{c},\mathbf{c}})^{-1}M^{c,t})}{\det(M^{c,t-1} - M^{t-1,c}(\mathbb{M}^{\mathbf{c},\mathbf{c}})^{-1}M^{c,t-1})}$$

- $(\mathbb{M}^{\mathbf{c},\mathbf{c}})^{-1}$ may still be computational expensive.
- However it can be achieved by repeatably applying block-inversion formula (cache in the memory):

$$M^{-1} = \begin{pmatrix} L & b \\ b^\top & c \end{pmatrix}^{-1} = \begin{pmatrix} L^{-1} + L^{-1}bb^\top L^{-1}d^{-1} & -L^{-1}bd^{-1} \\ -b^\top L^{-1}d^{-1} & d \end{pmatrix}^{-1}$$

# Computation at the Inverse

**Algorithm 1** Fast sampling for sequential DPPs

**Require:** $\{M_i\}_{i=1}^t$

$\{\mathbf{Y}_1^{(i)} \sim \det(M_{1,y_1^{(i)}})\}_{i=1}^N$ by (Kang 2013)

$\{W_1^{(i)} = 1/N\}_{i=1}^N$

**for** $k = 2, ..., t$ **do**

  $\{\widetilde{w}_k(y_{k-1}^{(i)}, y_k^{(i)}) = \det(L_{k,y_{k-1}^{(i)}}) / \det(L_{k-1,y_{k-1}^{(i)}})\}_{i=1}^N$

  $\{W_k^{(i)} = W_{k-1}^{(i)} \widetilde{w}_k^{(i)} / \sum_{i=1}^N (W_{k-1}^{(i)} \widetilde{w}_k^{(i)})\}_{i=1}^N$ [a]

  $\{y_k^{(i)} \sim K(y_{k-1}^{(i)}, y_k^{(i)})\}_{i=1}^N$

  $N_{k,ESS} = \{\sum_{i=1}^N (W_k^{(i)})^2\}^{-1}$

  **if** $N_{k,ESS} < \alpha \cdot N$ **then**

    $\{y_k^{(i)} \sim \text{Mult}(W_k^{(1)}, ..., W_k^{(N)})\}_{i=1}^N$

    $\{W_k^{(i)} = 1/N\}_{i=1}^N$

    move $\{y_k^{(i)}\}_{i=1}^N$ by $\pi_k$ invariant MCMC kernel $K_{\pi_k}(y_k^{(i)}, \cdot)$

  **end if**

**end for**

**return** $\{\{y_k^{(i)}, W_k^{(i)}\}_{i=1}^N\}_{k=1}^t$

[a] We use $W_k^{(i)}$ and $\widetilde{w}_k^{(i)}$ to replace $W_k(y_{1:k}^{(i)})$ and $\widetilde{w}_k(y_{k-1}^{(i)}, y_k^{(i)})$ for simplicity