EE120 - Fall'19 - Lecture 13 Notes[1]

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Review of the Sampling Theorem

Suppose we have samples of a continuous-time signal x taken every T units of time, that is, with the sampling frequency:

$$\omega_s = \frac{2\pi}{T}. \tag{1}$$

Can we reconstruct $x(t)$ from the samples $\{x(nT)\}_{n\in\mathbb{Z}}$ with an appropriate interpolation?

The Sampling Theorem by Shannon and Nyquist shows that the answer is yes if x is bandlimited, i.e.,

$$X(\omega) = 0 \quad |\omega| > \omega_M \tag{2}$$

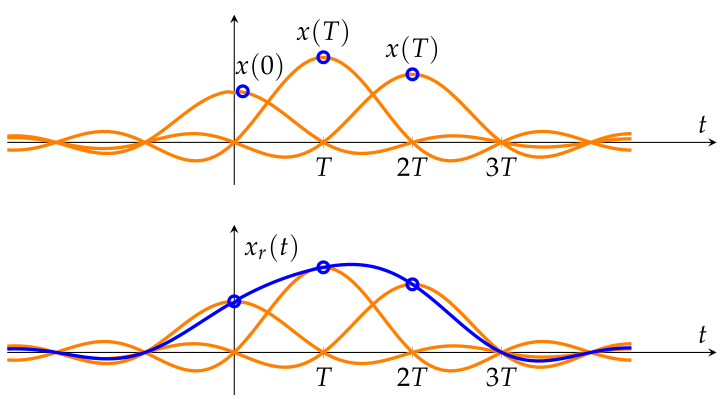
for some finite frequency ω_M, and the sampling frequency satisfies

$$\omega_s > 2\omega_M. \tag{3}$$

The proof of this theorem, stated more precisely below, uses scaled sinc functions for interpolation:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\mathrm{sinc}\left(\frac{t-nT}{T}\right). \tag{4}$$

Each sinc function in this sum is centered at a sample point. Verify that (4), when evaluated at a sample point, returns the value of x at that point. This is illustrated in the figure below.



<u>Sampling Theorem:</u> If x is bandlimited as in (2) and we select T such that the sampling frequency (1) satisfies (3), then $x_r(t) = x(t)$. □

Proof: We view x_r in (4) as the output of a LTI system with impulse response

$$h_r(t) = \text{sinc}\left(\frac{t}{T}\right) \tag{5}$$

when the input is

$$x_p(t) := \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT). \tag{6}$$

Show that the convolution of these two indeed gives x_r in (4). Thus,

$$X_r(\omega) = H_r(\omega)X_p(\omega) \tag{7}$$

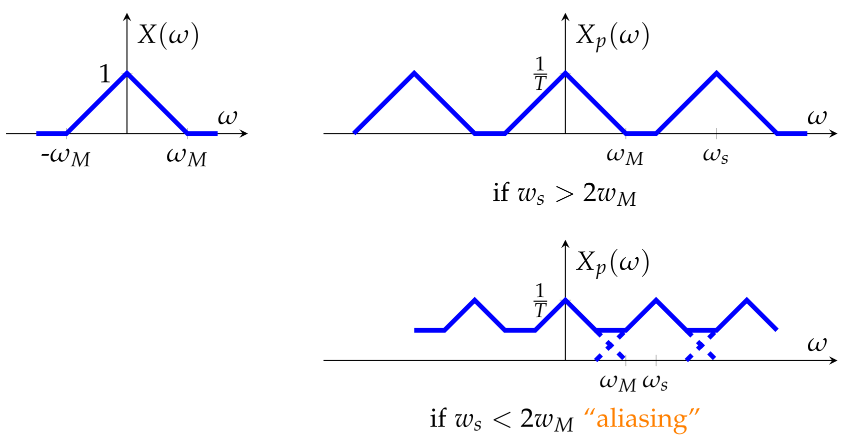
where

$$H_r(\omega) = \begin{cases} T & |\omega| < \frac{\pi}{T} \\ 0 & |\omega| > \frac{\pi}{T} = \frac{\omega_s}{2} \end{cases} \tag{8}$$

from the Fourier Transform of (5). Moreover, as shown last time,

$$X_p(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) \tag{9}$$

and (3) guarantees that the shifted copies in this sum do not overlap:



Thus, $X_p(\omega) = \frac{1}{T}X(\omega)$ when $|\omega| \leq \frac{\omega_s}{2}$, and (7) and (8) give

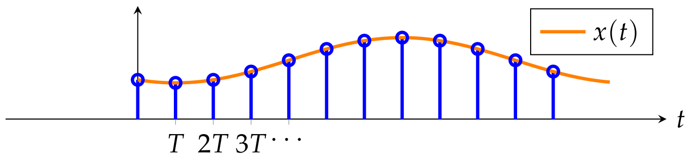
$$X_r(\omega) = X(\omega).$$

Fourier Transform of the Sampled Signal

We can view the samples of x as a discrete-time signal:

$$x_d[n] = x(nT) \tag{10}$$

as depicted below.



The DTFT of x_d is related to the CTFT of x_p in (6) by:

$$\boxed{X_d(e^{j\Omega})\Big|_{\Omega=\omega T} = X_p(\omega)} \tag{11}$$

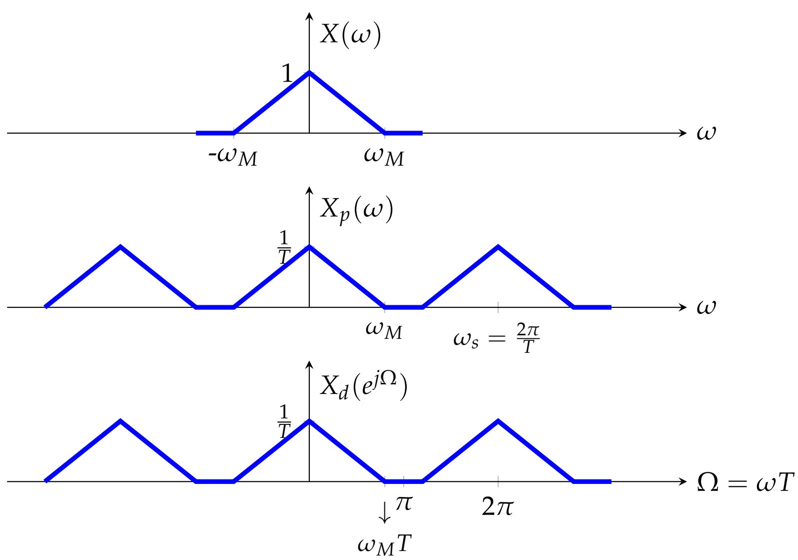
because

$$\begin{aligned} X_p(\omega) &= \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\omega T n} \\ X_d(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega n} \end{aligned} \qquad \begin{aligned} &\Omega = \omega T \\ &\omega : \text{radians/sec.} \\ &\Omega : \text{radians} \end{aligned}$$

Combining (11) with (9), we see that

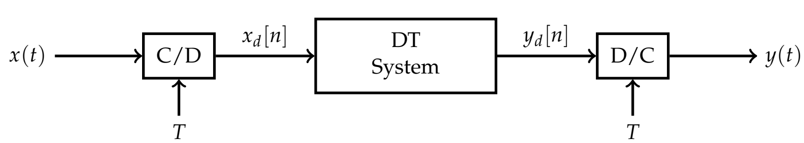
$$X_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega T - 2\pi k)$$

as depicted below:

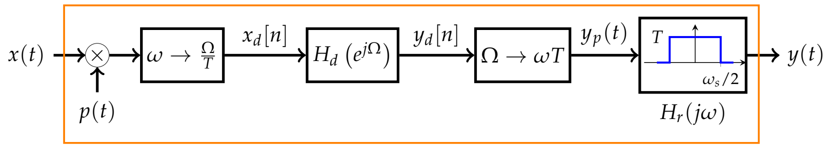


Discrete Time Processing of Continuous Time Signals

Section 7.4 *in* Oppenheim & Willsky



The combined system with the continuous-time input x and output y is linear but not time-invariant (to be shown in homework); thus, it doesn't have a well-defined frequency response $H(\omega)$. However, if x is bandlimited by $\frac{w_s}{2} = \frac{\pi}{/}T$, an "effective" $H(\omega)$ can be calculated:



$$Y_d(e^{j\Omega}) = H_d(e^{j\Omega})X_d(e^{j\Omega}) = H_d(e^{j\Omega})X_p(\Omega/T) \tag{12}$$

$$Y_p(\omega) = Y_d(e^{j\omega T}) = H_d(e^{j\omega T})X_p(\omega) \tag{13}$$

$$Y(\omega) = \begin{cases} TH_d(e^{j\omega T})X_p(\omega) & |\omega| < \omega_s/2 \\ 0 & |\omega| > \omega_s/2. \end{cases} \tag{14}$$

$$X_p(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) \tag{15}$$

Combining (14) and (15):

$$Y(\omega) = \begin{cases} H_d(e^{j\omega T})\displaystyle\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) & |\omega| < \omega_s/2 \\ 0 & |\omega| > \omega_s/2. \end{cases} \tag{16}$$

If x is bandlimited by $\omega_s/2$, no aliasing:

$$\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) = X(\omega) \qquad |\omega| < \omega_s/2 \tag{17}$$

$$Y(\omega) = \begin{cases} H_d(e^{j\omega T})X(\omega) & |\omega| < \omega_s/2 \\ 0 & |\omega| > \omega_s/2. \end{cases} \tag{18}$$

$$\boxed{H_{\text{eff}}(\omega) = \frac{Y(\omega)}{X(\omega)} = \begin{cases} H_d(e^{j\omega T}) & |\omega| < \omega_s/2 \\ 0 & |\omega| > \omega_s/2 \end{cases}} \tag{19}$$

This is the effective frequency response valid for inputs with bandwidth $< \omega_s/2$.