EE120 - Fall'19 - Lecture 12[1]

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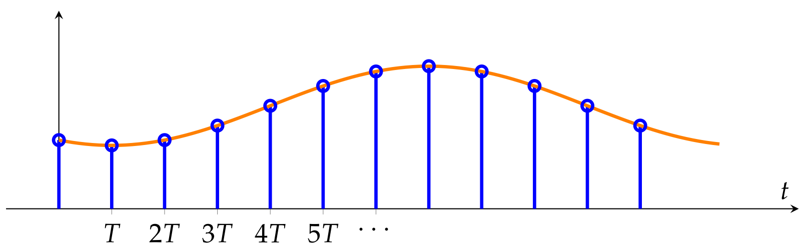
Sampling

Chapter 7 *in* Oppenheim & Willsky

Suppose we have samples of a continuous-time signal x taken every T units of time, that is, with the sampling frequency:

$$\omega_s = \frac{2\pi}{T}. \tag{1}$$

Can we reconstruct $x(t)$ from the samples $\{x(nT)\}_{n\in\mathbb{Z}}$ with an appropriate interpolation?



The Sampling Theorem by Shannon and Nyquist shows that the answer is yes if x is bandlimited, i.e.,

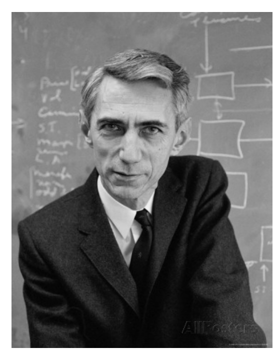
$$X(\omega) = 0 \quad |\omega| > \omega_M \tag{2}$$

for some finite frequency ω_M, and the sampling frequency satisfies

$$\omega_s > 2\omega_M. \tag{3}$$

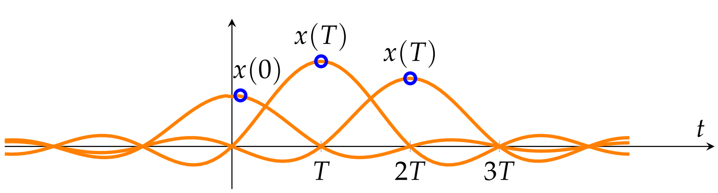
The proof of this theorem, stated more precisely below, uses scaled sinc functions for interpolation:

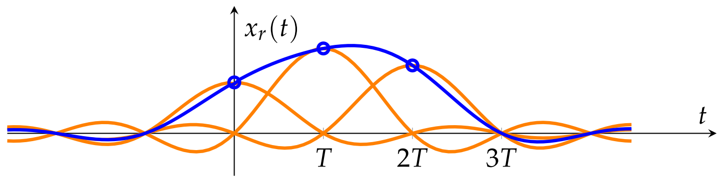
$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\mathrm{sinc}\left(\frac{t-nT}{T}\right). \tag{4}$$


Claude Shannon (1916-2001)


Harry Nyquist (1889-1976)

Each sinc function in this sum is centered at a sample point. Verify that (4), when evaluated at a sample point, returns the value of x at that point. This is illustrated in the figure below.





<u>Sampling Theorem:</u> If x is bandlimited as in (2) and we select T such that the sampling frequency (1) satisfies (3), then $x_r(t) = x(t)$.

Proof: We view x_r in (4) as the output of a LTI system with the following impulse response and input:

$$h_r(t) = \text{sinc}\left(\frac{t}{T}\right) \tag{5}$$

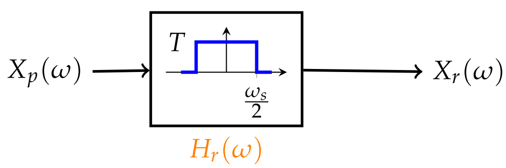
$$x_p(t) := \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT). \tag{6}$$

Show that the convolution of these two indeed gives x_r in (4). Thus,

$$X_r(\omega) = H_r(\omega)X_p(\omega) \tag{7}$$

where $H_r(\omega)$ is obtained from the Fourier Transform of (5):

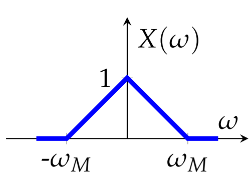
$$H_r(\omega) = \begin{cases} T & |\omega| < \frac{\pi}{T} \\ 0 & |\omega| > \frac{\pi}{T} = \frac{\omega_s}{2}. \end{cases} \tag{8}$$

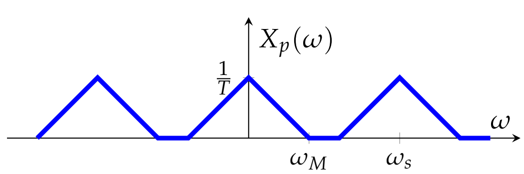


Moreover, as shown in the next section,

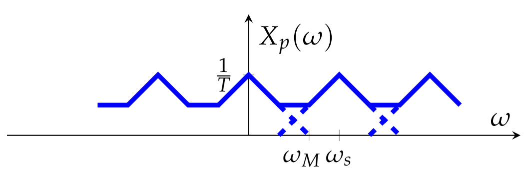
$$\boxed{X_p(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)} \tag{9}$$

and (3) guarantees that the shifted copies in this sum do not overlap:





if $w_s > 2w_M$



if $w_s < 2w_M$ "aliasing"

Thus, $X_p(\omega) = \frac{1}{T}X(\omega)$ when $|\omega| \leq \frac{\omega_s}{2}$, and (7) and (8) give

$$X_r(\omega) = X(\omega).$$

This means that the interpolated signal matches the continuous-time signal from which the samples were obtained. $\square$

There are two critical steps in the proof of the Sampling Theorem above. The first is to express the sinc interpolation in (4) as low pass filtering of x_p defined in (6). The second is to use (9), which states that the Fourier Transform of $X_p(\omega)$ consists of shifted copies of $X(\omega)$. When $\omega_M < \omega_s/2$, these copies do not overlap and the low-pass filter recovers x. When the copies overlap the low-pass filter will output a signal that differs from x, a phenomenon called "aliasing."

Impulse Train Sampling

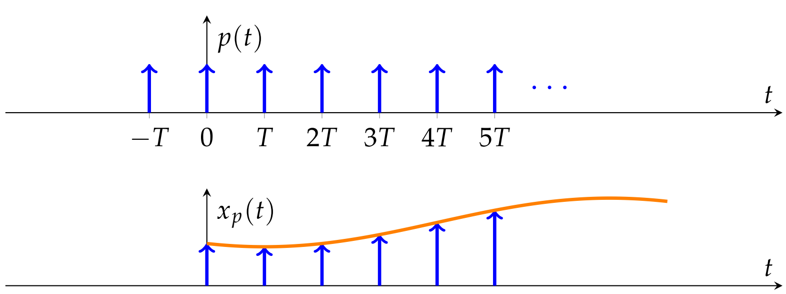
To see where (9) comes from, we note that x_p in (6) can be written as

$$x_p(t) = x(t) \cdot p(t) \tag{10}$$

where

$$p(t) := \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{11}$$

is called an "impulse train" and x_p is called the impulse train sampled signal. Both signals are illustrated below.



Recall that the Fourier series coefficients of the impulse train are $a_k = \frac{1}{T}$ for all k. Thus,

$$P(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_s) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s) \tag{12}$$

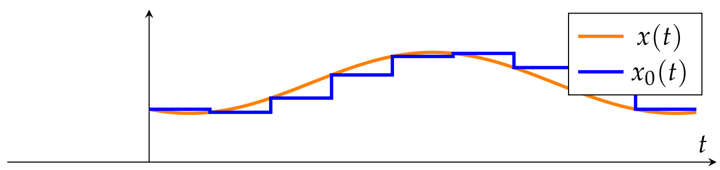
$$X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))d\theta = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)),$$

which proves (9).

Zero-order Hold Approximate Reconstruction

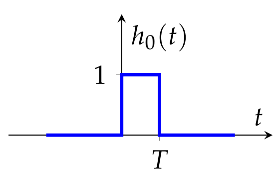
Since sinc interpolation corresponds to ideal low pass filtering, we next explore simpler forms of interpolation and compare the resulting frequency response to the ideal low-pass filter.

We first consider a zero-order hold which creates a staircase continuous-time signal with constant values in between the samples as shown below.



The resulting signal x_0 can be viewed as the output of a LTI system with impulse response h_0 shown below when the input is x_p:

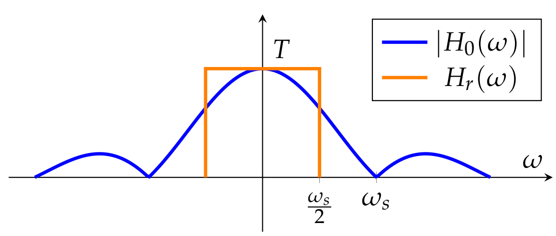
$$x_0(t) = (x_p * h_0)(t) \tag{13}$$



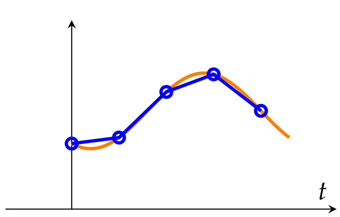
Thus,

$$H_0(\omega) = e^{-j\omega T/2} T\text{sinc}\left(\frac{T}{2\pi}\omega\right) \tag{14}$$

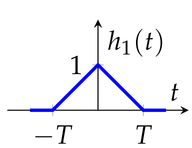
$$X_0(\omega) = H_0(\omega) X_p(\omega). \tag{15}$$



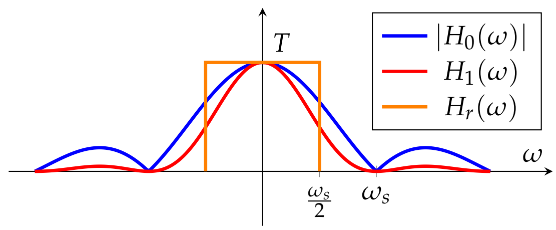
Linear Interpolation (First-order Hold)



$$x_1(t) = (x_p * h_1)(t)$$

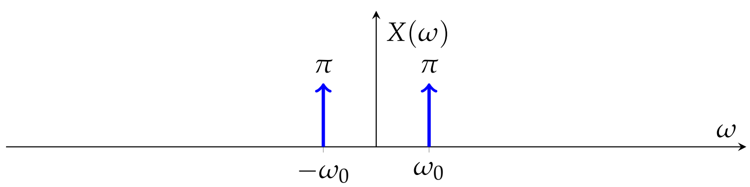


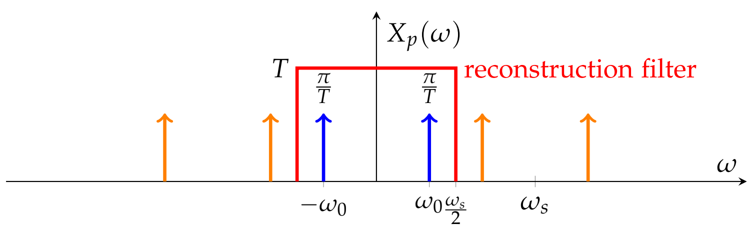
$$H_1(\omega) = T\text{sinc}^2\left(\frac{T}{2\pi}\omega\right) \tag{16}$$

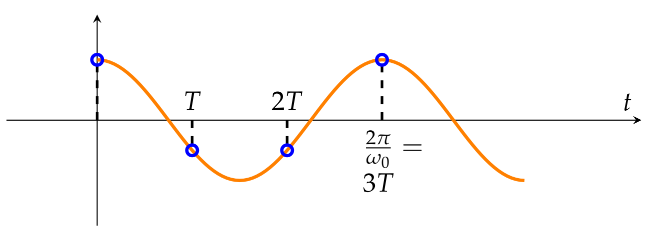


Simple Examples of Aliasing

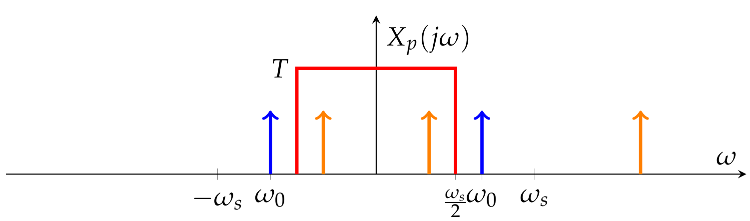
<u>Example 1</u>: $x(t) = \cos(\omega_0 t),\ \omega_s = 3\omega_0 > 2\omega_0$ (no aliasing)



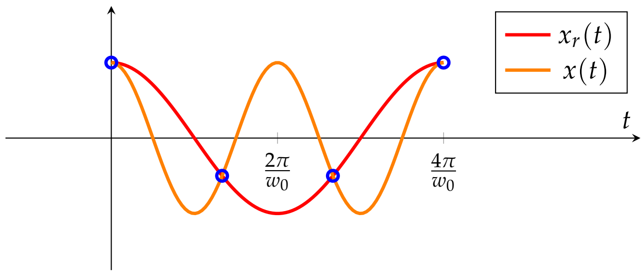




<u>Example 2</u>: $x(t) = \cos(\omega_0 t),\ \omega_s = \frac{3\omega_0}{2} < 2\omega_0$ (aliasing)

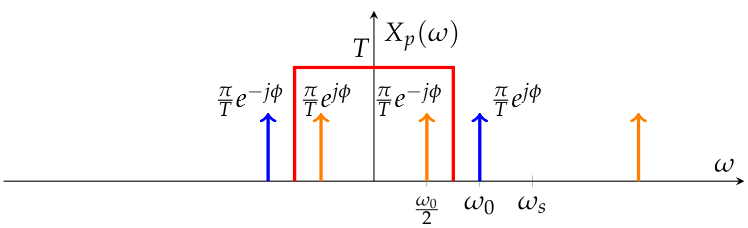


$$x_r(t) = \cos\left(\frac{\omega_0}{2}t\right) \neq x(t) \tag{17}$$



Example 3: (phase reversal)

$$x(t) = \cos(\omega_0 t + \phi) = \frac{1}{2}e^{j\phi}\, \underset{\substack{\updownarrow \\ 2\pi\delta(\omega-\omega_0)}}{e^{j\omega_0 t}} + \frac{1}{2}e^{-j\phi}e^{-j\omega_0 t}, \quad \omega_s = \frac{3\omega_0}{2} \tag{18}$$



$$X_r(\omega) = \underset{\substack{\updownarrow \\ \frac{1}{2\pi}e^{-j\frac{\omega_0}{2}t}}}{\pi e^{j\phi}\delta\left(\omega + \frac{\omega_0}{2}\right)} + \underset{\substack{\updownarrow \\ \frac{1}{2\pi}e^{j\frac{\omega_0}{2}t}}}{\pi e^{-j\phi}\delta\left(\omega - \frac{\omega_0}{2}\right)} \tag{19}$$

$$x_r(t) = \frac{1}{2}\left(e^{j\left(\frac{\omega_0}{2}t-\phi\right)} + e^{-j\left(\frac{\omega_0}{2}t-\phi\right)}\right) \tag{20}$$

$$= \cos\left(\frac{\omega_0}{2}t - \phi\right) \rightarrow \text{phase reversal} \tag{21}$$

$$= \cos\left(-\frac{\omega_0}{2}t + \phi\right) \tag{22}$$

Wagon wheel effect in movies: Wheel appears to rotate more slowly and in the opposite direction when actual speed exceeds half of the sampling rate (18-24 frames/second).

Example 4: Suppose we sample the signal

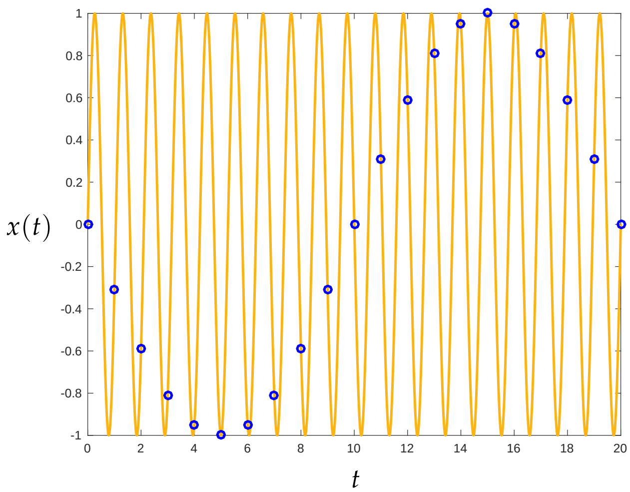
$$x(t) = \sin(1.9\pi t) \tag{23}$$

with $T = 1$ as shown in the figure below. It follows from Fourier analysis similar to the examples above that sinc interpolation gives[2]

[2] Show (24).

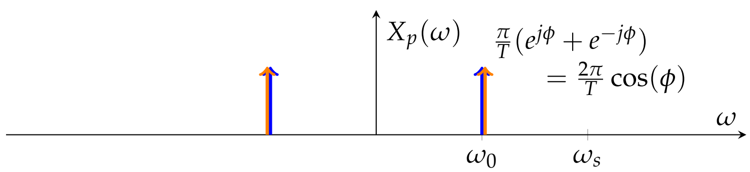
$$x_r(t) = \cos(0.1\pi t + \pi/2) = -\sin(0.1\pi t), \tag{24}$$

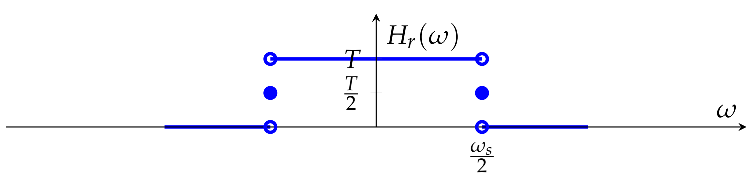
which is evident from the samples in the figure. The negative sign of $-\sin(0.1\pi x)$ is a result of the phase reversal discussed above.

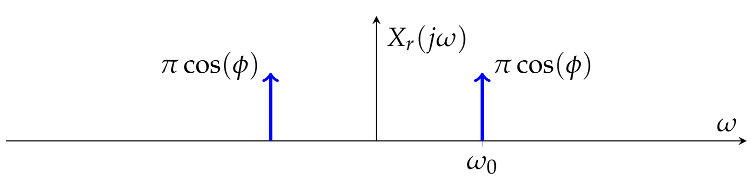


Example 5: (critical frequency)

$$x(t) = \cos(\omega_0 t + \phi) \qquad \omega_s = 2\omega_0 \tag{25}$$







$$x_r(t) = \cos(\phi)\cos(\omega_0 t) \neq x(t) \quad \text{unless } \phi = 0. \tag{26}$$