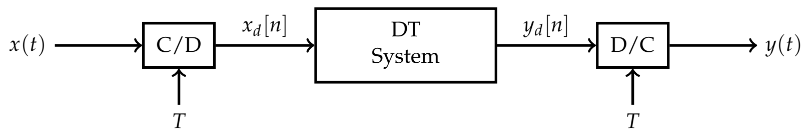
# *EE120 - Fall'19 - Lecture 14 Notes*[1]

*Murat Arcak*

*24 October 2019*



## *Discrete-Time Processing of Continuous-Time Signals*




Last time we saw that the combined system above mapping the continuous-time input $x$ to the continuous-time output $y$ is not LTI even if the discrete-time system at the core is LTI. Nevertheless, if $x$ is bandlimited by $\omega_s/2 = \pi/T$, then

$$Y(\omega) = H_d(e^{j\Omega})\Big|_{\Omega=\omega T} X(\omega) \tag{1}$$

where $H_d(e^{j\Omega})$ is the frequency response of the discrete-time system.

Example (Digital differentiator): Suppose we want $y(t)$ to approximate $\frac{dx(t)}{dt}$. We can implement the Euler approximation of the derivative in the discrete-time block:

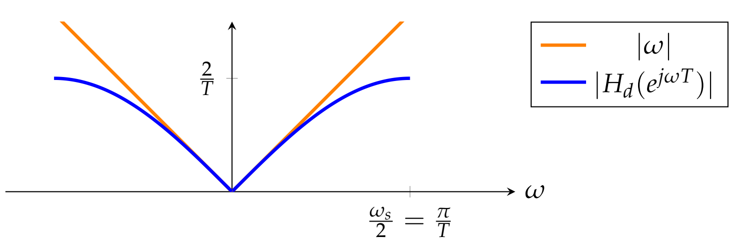
$$y_d[n] = \frac{x_d[n] - x_d[n-1]}{T} = h_d[n] * x_d[n] \tag{2}$$

where $h_d[0] = \frac{1}{T}$, $h_d[1] = \frac{-1}{T}$, $h_d[n] = 0$ for $n < 0$ and $n > 1$. Then,

$$H_d(e^{j\Omega}) = \sum_n h_d[n]e^{-j\Omega n} = \frac{1}{T}(1 - e^{-j\Omega}) \tag{3}$$

$$H_d(e^{j\Omega})\Big|_{\Omega=\omega T} = \frac{1}{T}(1 - e^{-j\omega T}). \tag{4}$$

The plot below shows close agreement for small frequencies between the magnitude of this effective frequency response and the frequency response $H(\omega) = j\omega$ of continuous time differentiation.

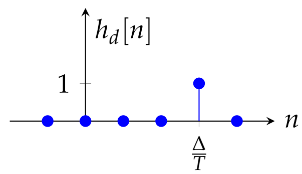



Example: Digital implementation of a delay: $y(t) = x(t - \Delta)$

How should we design the discrete-time system?

If $\Delta$ is an integer multiple of $T$, then

$$y_d[n] = x_d[n - \underbrace{\tfrac{\Delta}{T}}_{\text{integer}}] \quad \rightarrow$$




What if $\frac{\Delta}{T}$ is not an integer? We want $y(t) = x(t - \Delta)$, *i.e.*,

$$Y(\omega) = e^{-j\omega\Delta}X(\omega).$$

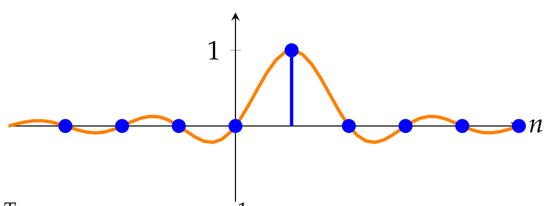
If we select $H_d(e^{j\Omega}) = e^{-j\frac{\Omega}{T}\Delta}$, $|\Omega| < \pi$, then the effective frequency response is indeed

$$H_d(e^{j\Omega})\Big|_{\Omega=\omega T} = e^{-j\omega\Delta}.$$

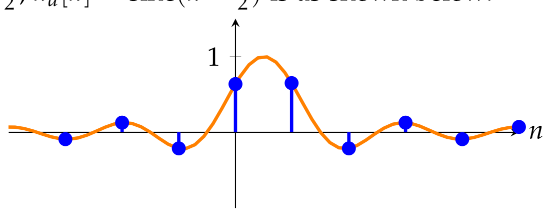
Thus, the desired impulse response of the discrete-time system is the inverse Fourier Transform of $H_d$:

$$h_d[n] = \text{sinc}\left(n - \frac{\Delta}{T}\right). \tag{5}$$

When $\Delta = T$, $h_d[n] = \text{sinc}(n-1)$ which is identical to $\delta[n-1]$:



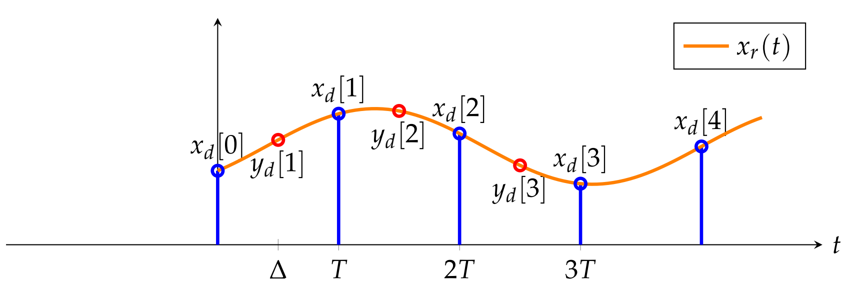

When $\Delta = \frac{T}{2}$, $h_d[n] = \text{sinc}(n - \frac{1}{2})$ is as shown below:




Note that

$$\begin{aligned} y_d[n] = (h_d * x_d)[n] = \sum_k x_d[k]h_d[n-k] &= \sum_k x(kT)\text{sinc}\left(n - \frac{\Delta}{T} - k\right) \\ &= x_r(t-\Delta)|_{t=nT} \end{aligned}$$

where $x_r(t)$ is the result of sinc interpolation as illustrated below.



## *Sampling of Discrete-Time Signals*

Section 7.5 *in* Oppenheim & Willsky

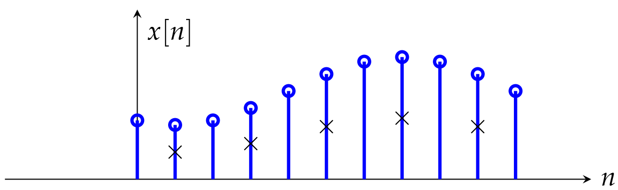
### *Impulse Train Sampling*

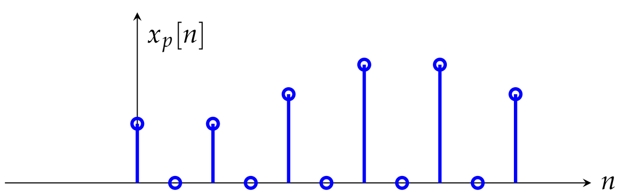
We obtain samples of a discrete-time signal $x[n]$ by multiplying it with a discrete-time impulse train with period $N$:

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN] \tag{6}$$

$$x_p[n] = x[n]p[n] = \sum_{k=-\infty}^{\infty} x[kN]\delta[n-kN]. \tag{7}$$

This is illustrated below for $N = 2$:

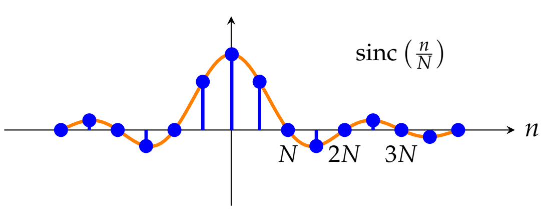


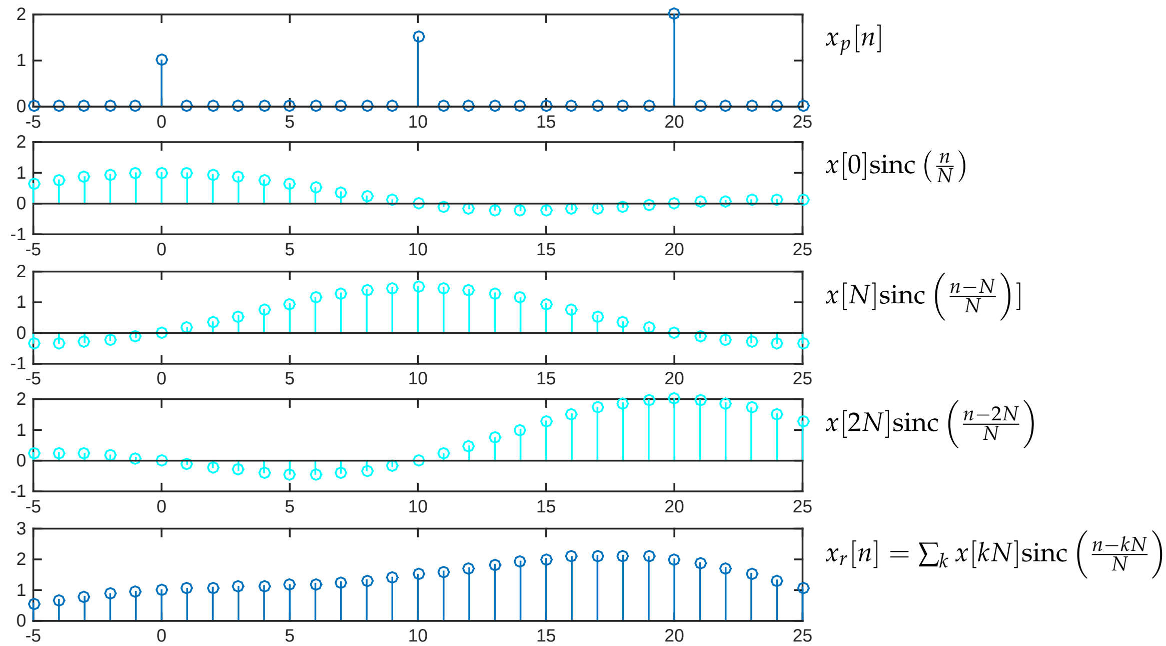

As in sampling of continuous-time signals we ask whether we can recover $x[n]$ from the samples with sinc interpolation:

$$x_r[n] = \sum_{k=-\infty}^{\infty} x_p[n]\text{sinc}\left(\frac{n-kN}{N}\right). \tag{8}$$

The sequence $\text{sinc}\left(\frac{n}{N}\right)$ is depicted below. The interpolated signal (8) is a sum of shifted copies of this sequence, each centered at a sample point and multiplied by the value of the sample at that point.




Sinc interpolation is illustrated below on a signal sampled with period $N = 10$.



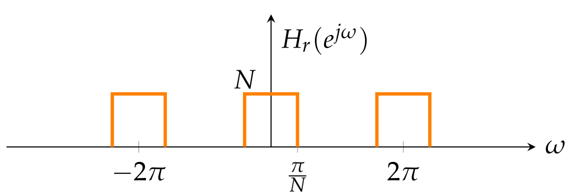

Note that we can view $x_r$ as the response of a LTI system with impulse response

$$h_r[n] = \text{sinc}\left(\frac{n}{N}\right) \tag{9}$$

to the input $x_p[n]$. Thus,

$$X_r(e^{j\omega}) = H_r(e^{j\omega})X_p(e^{j\omega}) \tag{10}$$

where $H_r(e^{j\omega})$ is as depicted below:




Moreover, as we will show later,

$$X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-k\omega_s)}) \quad \omega_s = \frac{2\pi}{N} \tag{11}$$

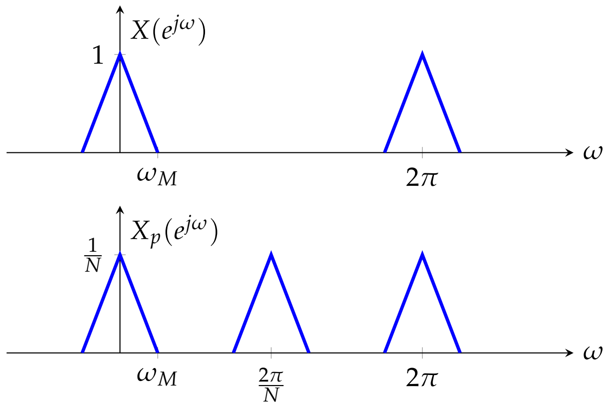
which consists of shifted copies of $X(e^{j\omega})$. Since these copies are $\omega_s$ apart, they do not overlap provided

$$X(e^{j\omega}) = 0 \quad \omega_M < |\omega| \leq \pi \tag{12}$$

and

$$\omega_s > 2\omega_M. \tag{13}$$

Below is an example of $X(e^{j\omega})$ and $X_p(e^{j\omega})$ when $N = 2$ and (13) holds:




Thus,

$$X_p(e^{j\omega}) = \frac{1}{N}X(e^{j\omega}) \quad |\omega| \leq \frac{\omega_s}{2} = \frac{\pi}{N} \tag{14}$$

and, from (10),

$$X_r(e^{j\omega}) = X(e^{j\omega}). \tag{15}$$

This leads to the following conclusion:

Discrete-Time Sampling Theorem: If $x[n]$ is bandlimited as in (12) and we select the sampling period $N$ such that $\omega_s = \frac{2\pi}{N}$ satisfies (13), then $x_r[n] = x[n]$. □

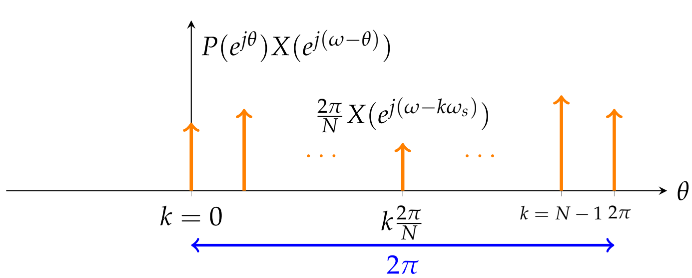
To see how (11) is obtained note that the Fourier series coefficients of the impulse train $p[n]$ are $a_k = \frac{1}{N}$ for all $k$. Therefore,

$$P(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\Big(\omega - k\underbrace{\frac{2\pi}{N}}_{=\omega_s}\Big) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s).$$

Since $x_p[n] = x[n]p[n]$, the multiplication property of DTFT implies:

$$X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})X(e^{j(\omega-\theta)})d\theta \tag{16}$$

where the integrand has the form:

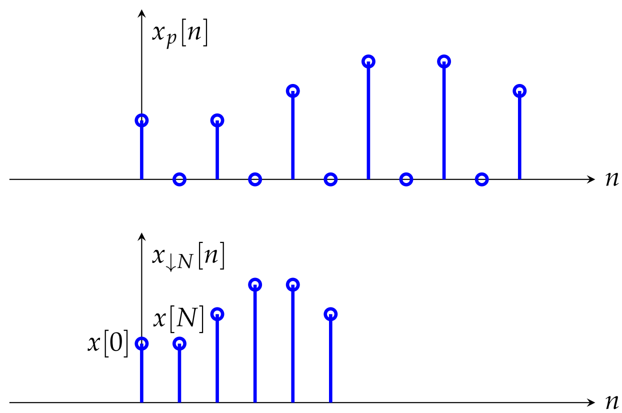



Integrating this over $\theta$ and dividing by $2\pi$ we get (11).

*Downsampling and Upsampling*

"Downsampling" a discrete-time sequence $x[n]$ means selecting every $N$th sample and discarding the rest. This is the same as removing the $N-1$ zeros between the samples in $x_p[n]$:

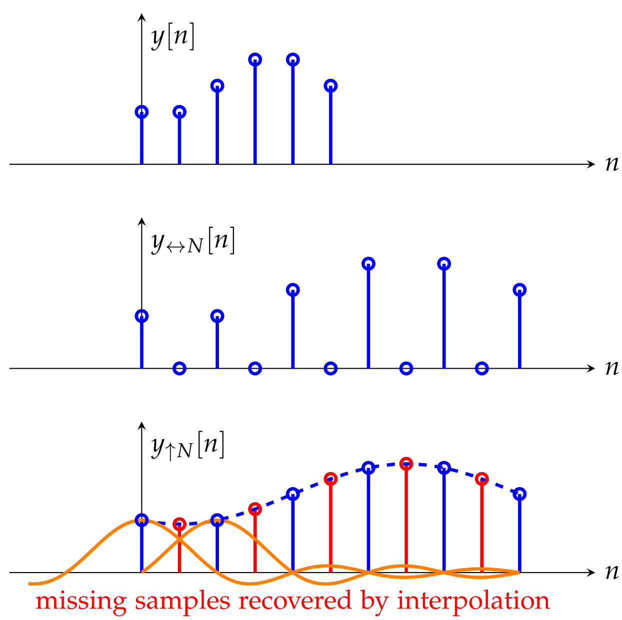
$$x_{\downarrow N}[n] = x[Nn] = x_p[Nn]. \tag{17}$$




To "upsample" a sequence $y[n]$ we do the opposite of downsampling. First we expand $y[n]$ by a factor of $N$ and pad zeros in between:

$$y_{\leftrightarrow N}[n] = \begin{cases} y[n/N] & n = 0, \mp N, \mp 2N, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Then we apply sinc interpolation to $y_{\leftrightarrow N}[n]$:




Note from the Discrete-Time Sampling Theorem that downsampling followed by upsampling recovers the original signal if (12)-(13) hold.

## 2D Sampling

Given $x(t_1, t_2)$ and sampling periods $T_1$, $T_2$:

$$x_d[n_1, n_2] \triangleq x(n_1T_1, n_2T_2).$$

Impulse train sampling:

$$x_p(t_1, t_2) = x(t_1, t_2)p(t_1, t_2)$$

where

$$p(t_1, t_2) \triangleq \sum_{n_1}\sum_{n_2} \delta(t_1 - n_1T_1, t_2 - n_2T_2).$$

2D CTFT gives:

$$X_p(\omega_1, \omega_2) = \frac{1}{T_1T_2} \sum_{k_1}\sum_{k_2} X\left(\omega_1 - k_1\omega_{s_1}, \omega_2 - k_2\omega_{s_2}\right)$$

where

$$\omega_{s_1} = \frac{2\pi}{T_1} \quad \text{and} \quad \omega_{s_2} = \frac{2\pi}{T_2}.$$

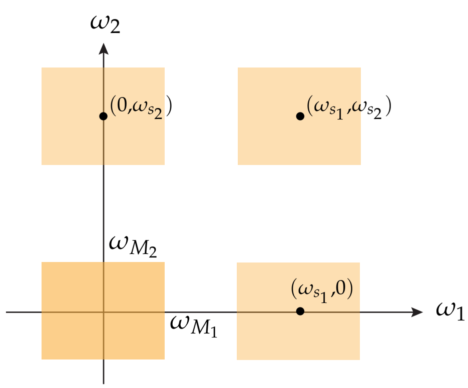
Therefore, if $x(t_1, t_2)$ is bandlimited:

$$X(\omega_1, \omega_2) = 0 \quad \text{when } |\omega_1| > \omega_{c_1} \text{ or } |\omega_2| > \omega_{c_2}$$

and

$$\omega_{s_1} > 2\omega_{M_1}, \qquad \omega_{s_2} > 2\omega_{M_2},$$

then there is no aliasing upon sampling:




Thus, $x(t_1, t_2)$ can be reconstructed from its samples with the sinc interpolation:

$$x_r(t_1, t_2) = \sum_{k=-\infty}^{\infty} x(n_1T_1, n_2T_2)\text{sinc}\left(\frac{t_1 - n_1T_1}{T_1}\right)\text{sinc}\left(\frac{t_2 - n_2T_2}{T_2}\right).$$

A discrete version of 2D sampling and interpolation can be derived similarly.