EE120 - Fall'19 - Lecture 2 Notes[1]

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Properties of Convolution

1. Unit impulse is the identity element:

$$\boxed{(x * \delta)[n] = x[n]}$$

Proof. Since $\delta[n-k] = 0$ for all k except for $k = n$, we have

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = (x[k]\delta[n-k])|_{k=n} = x[n]\delta[0] = x[n].$$

2. Convolution of a signal with a shifted impulse shifts the signal:

$$\boxed{x[n] * \delta[n-N] = x[n-N]}$$

Proof. Using the commutative property proven below,

$$x[n] * \delta[n-N] = \delta[n-N] * x[n] = \sum_{k=-\infty}^{\infty} \delta[k-N]x[n-k].$$

Since $\delta[k-N] = 0$ for all k except for $k = N$, the sum is

$$\delta[k-N]x[n-k]|_{k=N} = \delta[0]x[n-N] = x[n-N].$$

3. Commutative property:

$$\boxed{x * h = h * x}$$

Proof. The order of x and h doesn't matter because

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{r=-\infty}^{\infty} x[n-r]h[r] = (h * x)[n],$$

where we have used the change of variables $r := n - k$.

4. Distributive property:

$$\boxed{x * (h_1 + h_2) = x * h_1 + x * h_2}$$

Proof. This follows because

$$\sum_{k=-\infty}^{\infty} x[k](h_1[n-k] + h_2[n-k]) = \sum_{k=-\infty}^{\infty} x[k]h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k]h_2[n-k].$$

5. Associative property:

$$\boxed{x * (h_1 * h_2) = (x * h_1) * h_2} \tag{1}$$

Proof. The left-hand side of (1) is equal to

$$\sum_{k=-\infty}^{\infty} x[n-k] \underbrace{\sum_{r=-\infty}^{\infty} h_1[r]h_2[k-r]}_{(h_1 * h_2)[k]} = \sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} x[n-k]h_1[r]h_2[k-r].$$

If we define new variables $m := n-k$ and $s := n+r-k$, then $r = s-m$ and $k-r = n-s$, and we can rewrite the sum above as

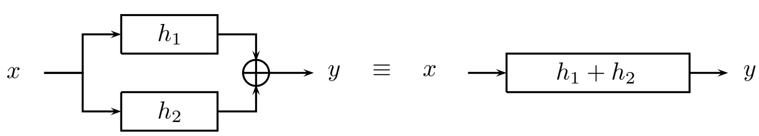
$$\sum_{s=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x[m]h_1[s-m]h_2[n-s] = \sum_{s=-\infty}^{\infty} \underbrace{\left(\sum_{m=-\infty}^{\infty} x[m]h_1[s-m]\right)}_{=(x*h_1)[s]} h_2[n-s]$$

which is the right-hand side of (1).

Back to LTI Systems

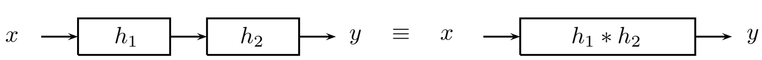
The properties of convolution discussed above have important implications for LTI systems:

- The distributive property implies that the parallel combination of two LTI systems with impulse responses h_1 and h_2 can be represented as an equivalent LTI system with impulse response $h_1 + h_2$:



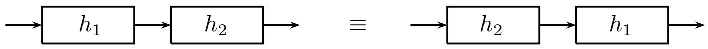
This is because the block diagram on the left produces the output $x * h_1 + x * h_2$ and, by the distributive property, this is equal to $x * (h_1 + h_2)$, which is interpreted on the right as the output of a LTI system with impulse response $h_1 + h_2$.

- Likewise, the associative property implies that we can combine the series interconnection of two LTI systems with impulse responses h_1 and h_2 into an equivalent LTI system with impulse response $h_1 * h_2$:



Indeed, the block diagram on the left produces the output $(x * h_1) * h_2$ and the one on the right produces $x * (h_1 * h_2)$.

- Combining the observation above with the commutative property, we conclude that swapping two LTI systems in a series interconnection results in an identical system:



Determining Causality and Stability from the Impulse Response

Since the impulse response fully characterizes a LTI system, we can judge the causality and stability properties from the impulse response alone:

- A discrete-time LTI system is causal if and only if:

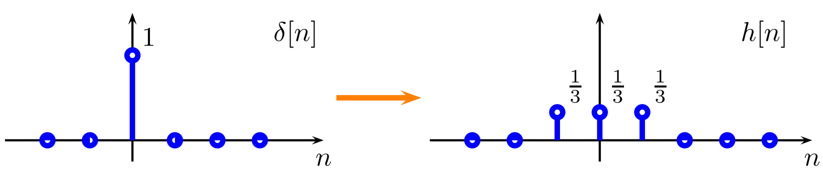
$$\boxed{h[n] = 0 \text{ for all } n < 0.} \tag{2}$$

Proof. Since $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$, if $h[k] = 0$ for all $k < 0$, then $y[n]$ depends on $x[n-k]$, where $k > 0$. Thus only present and past values of the input affect the output. Conversely, if $h[k] \neq 0$ for some $k < 0$, then $y[n]$ depends on $x[n-k]$, which is a future value of the input since $k < 0$.

Example: The moving average system:

$$y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right)$$

is non-causal, since $y[n]$ depends on $x[n+1]$. The impulse response shown below confirms non-causality, as $h[-1] \neq 0$.



Example: The impulse response of the accumulator system, defined for $n \geq 0$ by

$$y[n] - y[n-1] = x[n], \; y[-1] = 0, \tag{3}$$

is the unit step, $u[n]$. Causality follows because $u[n] = 0$ for all $n < 0$.

- A discrete-time LTI system is stable if and only if:

$$\boxed{\sum_{k=-\infty}^{\infty} |h[k]| < \infty.} \tag{4}$$

Proof. "If and only if" means that (4) is both necessary and sufficient for stability, which we prove separately:

Sufficiency: Suppose $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ and show that bounded inputs give bounded outputs:

$|x[n]| \leq B$ for all n, for some $B > 0$.

$|y[n]| = |\sum_k x[n-k]h[k]| \leq \sum_k |x[n-k]| \cdot |h[k]| \leq B \sum_k |h[k]| < \infty.$

Necessity: To prove "stable $\Rightarrow \sum_k |h[k]| < \infty$" prove the contrapositive:

$$"\sum_k |h[k]| = \infty \Rightarrow \text{ unstable."} \tag{5}$$

Let $x[n] = \text{sign}\{h[-n]\}$. Then, since $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$:

$$y[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_k h[k]\text{sign}\{h[k]\} = \sum_k |h[k]| = \infty. \tag{6}$$

Example: The moving average system above is stable because

$$\sum_k |h[k]| = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1 < \infty. \tag{7}$$

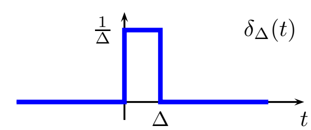
Example: The accumulator system is unstable because its impulse response is the unit response, for which $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$.

Continuous-Time LTI Systems: Convolution Integral

Section 2.2 *in* Oppenheim & Willsky

In continuous-time the unit impulse is defined as:

$$\delta(t) := \lim_{\Delta \to 0} \delta_\Delta(t) \tag{8}$$



where $\delta_\Delta(t)$ is a pulse with width Δ and amplitued $1/\Delta$, as shown on the right. Note that the width goes to zero and the amplitude goes to ∞ as $\Delta \to 0$, but the area underneath remains equal to one.

Although the limit in (8) does not define a function in the strict sense, the Theory of Distributions to be discussed briefly in Lecture 8 justifies its use along with the properties:

$$f(t)\delta(t) = f(0)\delta(t) \quad \text{and} \quad f(t)\delta(t-T) = f(T)\delta(t-T) \tag{9}$$

for any function f.

Let $h(t)$ denote the response of a LTI system to $\delta(t)$. Then, for any input $x(t)$, the output is :

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad \text{"convolution integral"}} \tag{10}$$

Proof. First, note that the staircase approximation in Figure 1 recovers $x(t)$ as $\Delta \to 0$:

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta\delta_\Delta(t - k\Delta). \tag{11}$$

Next, let $h_\Delta(t)$ denote the response of the system to $\delta_\Delta(t)$ and note from the LTI property that the response to each term in the sum above is $x(k\Delta)\Delta h_\Delta(t-k\Delta)$. Thus, the response to $x(t)$ is

$$y(t) = \lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t-k\Delta)\Delta = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau. \quad (12)$$

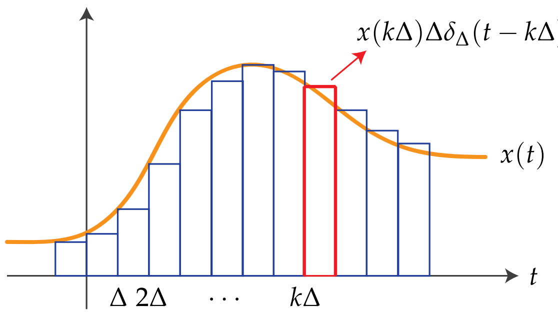


Figure 1: Staircase approximation of $x(t)$.

The convolution integral possesses similar properties to the discrete-time convolution sum. In particular, it follows from the definition of the convolution integral above combined with (9) that:

1. $(x * \delta)(t) = x(t)$
2. $x(t) * \delta(t-T) = x(t-T)$.

The commutative, distributive, and associative properties also hold. Therefore, the observations for parallel and series connections of LTI systems on page 2 hold in continuous-time as well.

Similarly, the causality of a continuous-time LTI system is equivalent to the property:

$$\boxed{h(t) = 0 \text{ for all } t < 0,} \quad (13)$$

and the stability criterion is:

$$\boxed{\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty.} \quad (14)$$