EE120 - Fall'19 - Lecture 9 Notes[1]

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Discrete Time Fourier Transform (DTFT) Continued

Conjugation and Conjugate Symmetry Property

$$\boxed{x^*[n] \longleftrightarrow X^*(e^{-j\omega})} \tag{1}$$

Thus, if x is real-valued, then:

$$X(e^{j\omega}) = X^*(e^{-j\omega}) \tag{2}$$

or using the periodicity of $X(e^{j\omega})$:

$$X(e^{j\omega}) = X^*(e^{j(2\pi-\omega)}) \quad \Rightarrow \quad |X(e^{j\omega})| = |X(e^{j(2\pi-\omega)})| \tag{3}$$

$$\measuredangle X(e^{j\omega}) = -\measuredangle X(e^{j(2\pi-\omega)}) \tag{4}$$

Combining this with time reversal property $x[-n] \;\longleftrightarrow\; X(e^{-j\omega})$:

$$x[-n] \;\longleftrightarrow\; X(e^{-j\omega}) = X^*(e^{j\omega}) \quad \text{when } x \text{ is real.} \tag{5}$$

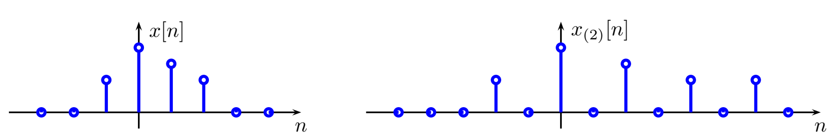
Thus, for an even symmetric signal ($x[n] = x[-n]$ for all n) we conclude $X(e^{j\omega}) = X^*(e^{j\omega})$, i.e., $X(e^{j\omega})$ is real. Likewise, for an odd symmetric signal ($x[n] = -x[-n]$ for all n), $X(e^{j\omega}) = -X^*(e^{j\omega})$, i.e., $X(e^{j\omega})$ is purely imaginary.

Time Expansion

Define the expanded signal

$$\boxed{x_{(M)}[n] = \begin{cases} x[n/M] & \text{if } n = 0, \mp M, \mp 2M, \ldots \\ 0 & \text{otherwise} \end{cases}} \tag{6}$$

illustrated below:

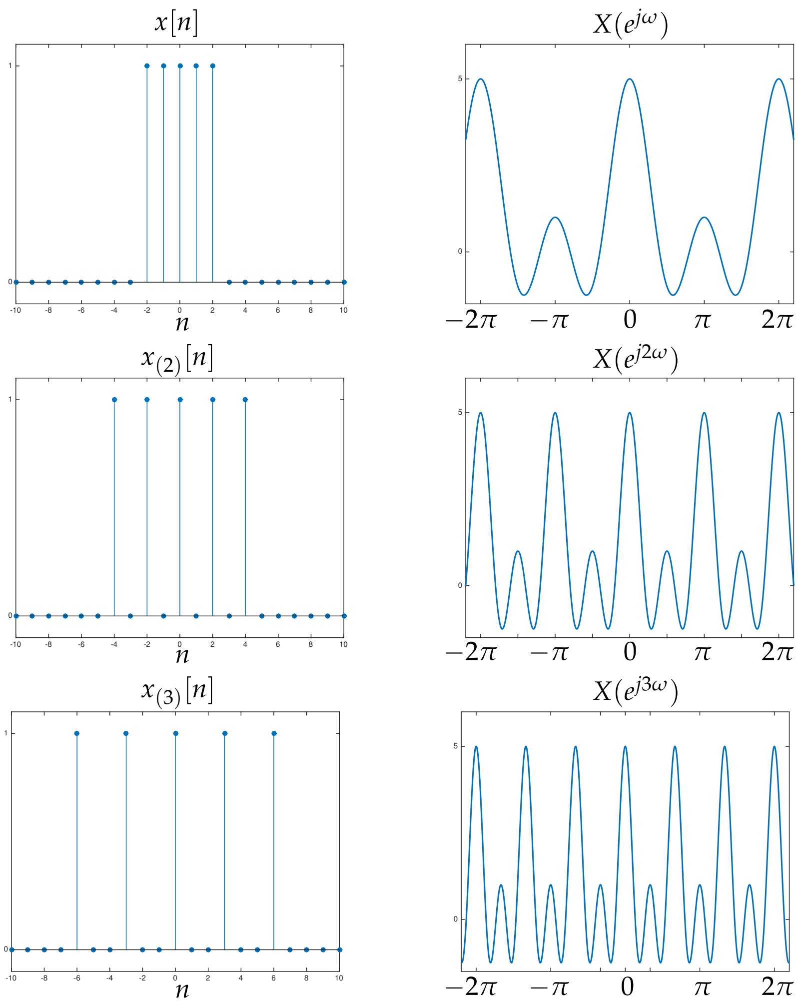


Then

$$\boxed{x_{(M)}[n] \longleftrightarrow X(e^{j\omega M}).} \tag{7}$$

Proof:
$$\sum_{n=-\infty}^{\infty} x_{(M)}[n]e^{-j\omega n} = \sum_{k=-\infty}^{\infty} \underbrace{x_{(M)}[kM]}_{=x[k]} e^{-j\omega kM}$$
$$= \sum_{k=-\infty}^{\infty} x[k]e^{-j(\omega M)k} = X(e^{j\omega M}).$$

See figure below for an illustration on a rectangular pulse (top) expanded with $M = 2$ (middle) and $M = 3$ (bottom). The Fourier transforms shown on the right obey the property (7).



We can compare the expansion factor M to $1/a$ in the continuous time property $x(at) \longleftrightarrow \frac{1}{|a|}X(j\frac{\omega}{a})$.

Differentiation in Frequency

$$\boxed{nx[n] \longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega}} \quad (8)$$

Proof:
$$\begin{aligned} X(e^{j\omega}) &= \sum_n x[n]e^{-j\omega n} \\ \frac{dX(e^{j\omega})}{d\omega} &= -j\sum_n nx[n]e^{-j\omega n} \end{aligned}$$

Multiply both sides by j and substitute $-j^2 = 1$.

The continuous time analog to (8) is: $tx(t) \longleftrightarrow j\frac{dX(j\omega)}{d\omega}$

The differentiation in time property $\frac{dx(t)}{dt} \leftrightarrow j\omega X(j\omega)$ has no direct discrete-time counterpart, but we can compare it to:

$$\boxed{x[n] - x[n-1] \longleftrightarrow (1 - e^{-j\omega})X(e^{j\omega})} \tag{9}$$

which follows from the time shift property.

Parseval's Relation

$$\boxed{\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega} \tag{10}$$

Multiplication Property

Section 5.5 *in* Oppenheim & Willsky

$$\boxed{x_1[n]x_2[n] \quad \longleftrightarrow \quad \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta} \tag{11}$$

The operation on the right is called "periodic convolution" because the integrand is periodic and integration is over one period.

Proof: Apply the synthesis equation to the right-hand side:

$$\begin{aligned} & \frac{1}{2\pi}\int_{2\pi} \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta e^{j\omega n} d\omega \\ = & \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta}) \underbrace{\frac{1}{2\pi}\int_{2\pi} X_2(e^{j(\omega-\theta)})e^{j\omega n} d\omega}_{= e^{j\theta n}x_2[n]} d\theta \\ = & x_2[n] \underbrace{\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n} d\theta}_{= x_1[n]} = x_1[n]x_2[n]. \end{aligned}$$

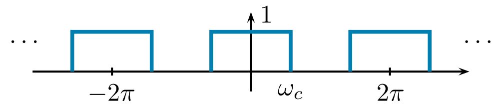
The integral over ω in the second line has the form of a synthesis equation applied to $X_2(e^{j(\omega-\theta)})$ and gives $e^{j\theta n}x_2[n]$ because of the frequency shift property $e^{j\theta n}x_2[n] \longleftrightarrow X_2(e^{j(\omega-\theta)})$.

<u>Example:</u> Use the multiplication property to find the DTFT of

$$x[n] = \underbrace{\frac{3}{4}\mathrm{sinc}\left(\frac{3}{4}n\right)}_{=: x_1[n]} \cdot \underbrace{\frac{1}{2}\mathrm{sinc}\left(\frac{1}{2}n\right)}_{=: x_2[n]}.$$

First note that:

$$\frac{\omega_c}{\pi}\text{sinc}\left(\frac{\omega_c}{\pi}n\right) \quad \longleftrightarrow$$



which follows by applying the synthesis equation to the DTFT shown on the right:

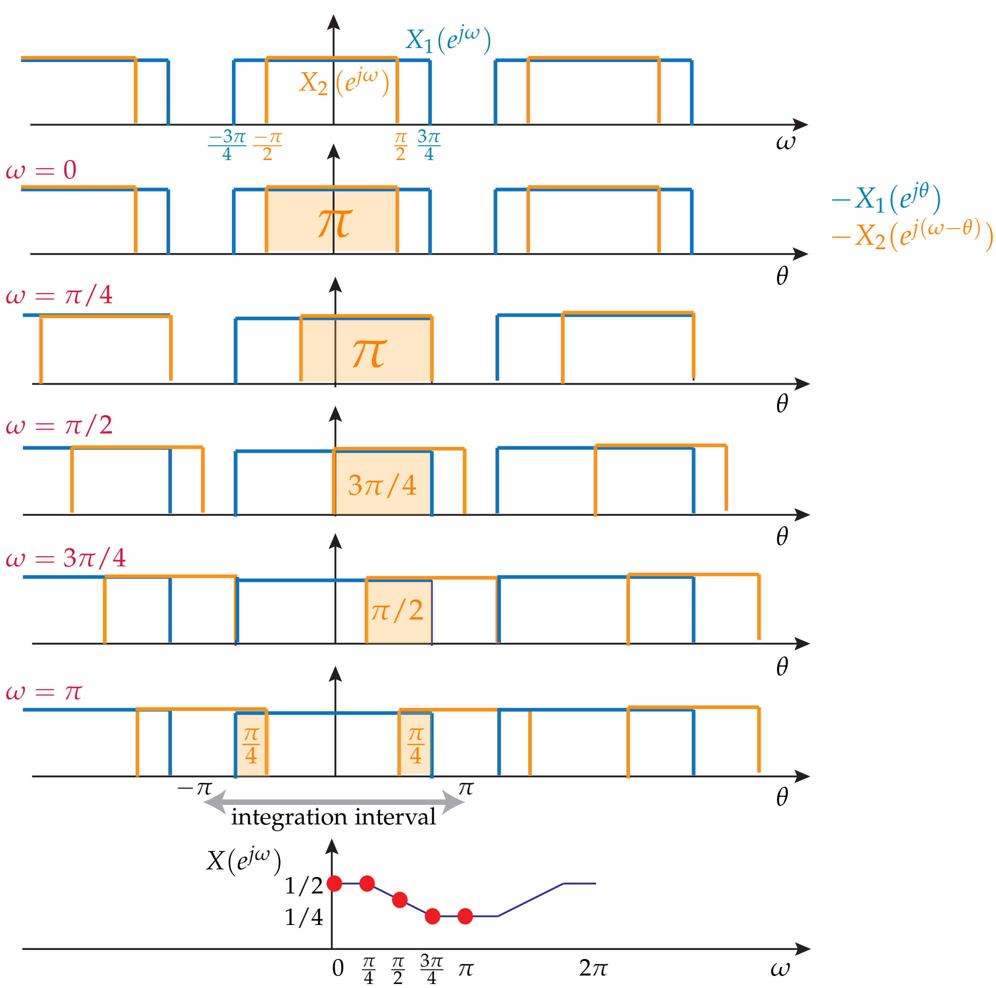
$$\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega = \frac{1}{2\pi}\frac{1}{jn}e^{j\omega n}\Big|_{-\omega_c}^{\omega_c} = \frac{1}{\pi n}\underbrace{\frac{1}{2j}(e^{j\omega_c n} - e^{-j\omega_c n})}_{=\sin\omega_c n}.$$

Thus $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$ are rectangular pulses in the frequency domain with cutoff frequencies $\frac{3\pi}{4}$ and $\frac{\pi}{2}$.

Then, by the multiplication property,

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta \tag{12}$$

which is a periodic convolution of these two pulses as illustrated below:



Convolution Property of DTFT

Section 5.4 *in* Oppenheim & Willsky

$$\boxed{(x_1 * x_2)[n] \longleftrightarrow X_1(e^{j\omega})X_2(e^{j\omega})} \tag{13}$$

Example: $x_1[n] = \alpha^n u[n] \quad |\alpha| < 1 \quad \longleftrightarrow \quad X_1(e^{j\omega}) = \dfrac{1}{1-\alpha e^{-j\omega}}$

$x_2[n] = \beta^n u[n] \quad |\beta| < 1 \quad \longleftrightarrow \quad X_2(e^{j\omega}) = \dfrac{1}{1-\beta e^{-j\omega}}$

$$X_1(e^{j\omega})X_2(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}$$

If $\alpha \neq \beta$, we use the partial fraction expansion:

$$X_1(e^{j\omega})X_2(e^{j\omega}) = \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}, \tag{14}$$

with $A + B = 1$ and $A\beta + B\alpha = 0$. This means $A = \frac{\alpha}{\alpha-\beta}$, $B = \frac{-\beta}{\alpha-\beta}$, and the inverse DTFT of (14) is

$$\begin{aligned}(x_1 * x_2)[n] &= \frac{\alpha}{\alpha-\beta}\alpha^n u[n] - \frac{\beta}{\alpha-\beta}\beta^n u[n] \\ &= \frac{1}{\alpha-\beta}\left(\alpha^{n+1} - \beta^{n+1}\right) u[n].\end{aligned}$$

When $\alpha = \beta$ we note:

$$X_1(e^{j\omega})X_2(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})^2} = \frac{e^{j\omega}}{\alpha} j \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) \tag{15}$$

and recall from the differentiation in frequency property that:

$$n x_1[n] \longleftrightarrow j\frac{dX_1(e^{j\omega})}{d\omega} = j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right).$$

Next we use the time shift property:

$$(n+1)x_1[n+1] \longleftrightarrow (e^{j\omega})j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

and note that dividing this DTFT by α recovers (15). Thus, the inverse DTFT of $X_1(e^{j\omega})X_2(e^{j\omega})$ is

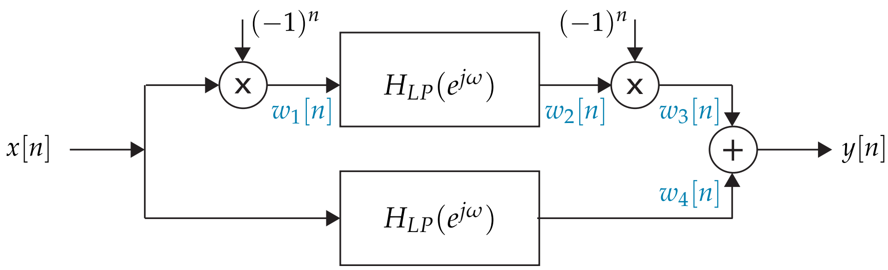
$$(x_1 * x_2)[n] = \frac{1}{\alpha}(n+1)x_1[n+1].$$

Finally, substituting $x_1[n] = \alpha^n u[n]$, we get

$$(x_1 * x_2)[n] = \frac{1}{\alpha}(n+1)\alpha^{n+1}u[n+1] = (n+1)\alpha^n u[n+1] = (n+1)\alpha^n u[n]$$

where we replaced $u[n+1]$ with $u[n]$ since $(n+1)\alpha^n = 0$ for $n = -1$.

Example: Determine the function performed by the block diagram below where H_{LP} is a low-pass filter with cutoff frequency $\omega_c < \pi/2$.

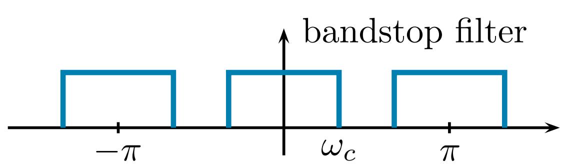


Using the frequency shift and convolution properties:

$$\begin{aligned}
W_1(e^{j\omega}) &= X(e^{j(\omega-\pi)}) \\
W_2(e^{j\omega}) &= H_{LP}(e^{j\omega})X(e^{j(\omega-\pi)}) \\
W_3(e^{j\omega}) &= H_{LP}(e^{j(\omega-\pi)})X(e^{j\omega}) \\
W_4(e^{j\omega}) &= H_{LP}(e^{j\omega})X(e^{j\omega}).
\end{aligned}$$

Then, adding $W_3(e^{j\omega})$ and $W_4(e^{j\omega})$:

$$Y(e^{j\omega}) = \underbrace{(H_{LP}(e^{j\omega}) + H_{LP}(e^{j(\omega-\pi)}))}X(e^{j\omega}) \tag{16}$$



Finding the Frequency Response from a Difference Equation

Section 5.8 *in* Oppenheim & Willsky

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{17}$$

Substitute $x[n] = \delta[n]$, and $y[n] = h[n]$:

$$\sum_{k=0}^{N} a_k h[n-k] = \sum_{k=0}^{M} b_k \delta[n-k] \tag{18}$$

Take the Fourier Transform of both sides (recall that $\delta[n] \leftrightarrow 1$):

$$\left(\sum_{k=0}^{N} a_k e^{-j\omega k}\right) H(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-j\omega k} \tag{19}$$

$$\boxed{H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}} \tag{20}$$

We can find the impulse response $h[n]$ from the inverse Fourier transform of $H(e^{j\omega})$:

Example:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n] \tag{21}$$

Frequency response:

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}} = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}$$

Partial fraction expansion:

$$H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}$$

Thus, the impulse response is:

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$

Example: Describe the LTI system with impulse response $h[n] = \alpha^n u[n]$, $|\alpha| < 1$, with a difference equation.

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \tag{22}$$

which is (20) with $b_0 = 1$, $\alpha_0 = 1$, and $a_1 = -\alpha$. Thus,

$$y[n] - \alpha y[n-1] = x[n].$$