EE120 - Fall'19 - Lecture 8 Notes[1]

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Continuous Time Fourier Transform (CTFT) Continued

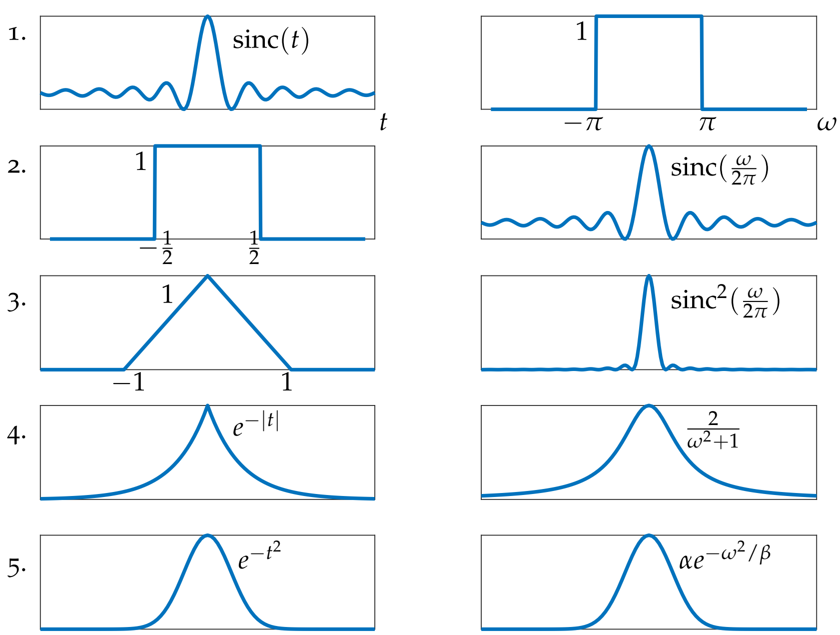
Convergence of the Fourier Integral

The theorem below provides a (simple but conservative) sufficient condition for the Fourier Transform to exist.

Theorem. If $\int_{-\infty}^{\infty} |x(t)|dt < \infty$ then $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ exists and is continuous. In addition $X(\omega) \to 0$ as $\omega \to \pm\infty$.

In Examples 2,3,4,5 below $x(t)$ satisfies the absolute integrability condition $\int_{-\infty}^{\infty} |x(t)|dt < \infty$ and $X(\omega)$ has the properties stated in the Theorem. By contrast, $x(t) = \text{sinc}(t)$ in Example 1 is not absolutely integrable[2]. Nevertheless the integral $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ converges. We will not prove this here, but we use this example to point out that the absolute integrability is only sufficient and not necessary.

[2] To see this assume to the contrary that it is absolutely integrable. Then the theorem states that $X(\omega)$ is continuous, which is a contradiction since $X(\omega)$ has jump discontinuities at $\omega \pm \pi$.



For other signals such, as $x(t) = 1$ or $x(t) = \cos(\omega_0 t)$, the Fourier integral does not converge; therefore, the Fourier Transform does not exists in the strict sense. However, a generalized[3] notion of Fourier Transform allows us to define Fourier Transforms for these functions

[3] This generalization is based on the Theory of Distributions which I will discuss briefly in class.

using the Dirac δ, which we called the unit impulse in Lecture 2 and defined as the limit of a sequence of functions[4].

[4] The limit is not a function in the strict sense but well defined as a distribution, as I will also discuss in class.

In particular

$$x(t) = 1 \quad \stackrel{FT}{\leftrightarrow} \quad X(\omega) = 2\pi\delta(\omega) \tag{1}$$

which we justify with the synthesis equation

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)e^{j\omega t}d\omega = 1.$$

The dual to this pair is

$$x(t) = \delta(t) \quad \stackrel{FT}{\leftrightarrow} \quad X(\omega) = 1, \tag{2}$$

as justified with the analysis equation:

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = 1.$$

A generalization of (1) gives:

$$x(t) = e^{j\omega_0 t} \quad \stackrel{FT}{\leftrightarrow} \quad X(\omega) = 2\pi\delta(\omega - \omega_0) \tag{3}$$

which can be justified with the synthesis equation

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}.$$

When $\omega_0 = 0$ we recover (1).

Fourier Transform of Periodic Signals

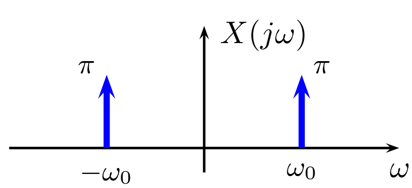
Section 4.2 *in* Oppenheim & Willsky

Using (3) and linearity, we can now define a generalized Fourier Transform for periodic signals expressed as Fourier Series:

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \stackrel{FT}{\leftrightarrow} \quad \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \tag{4}$$

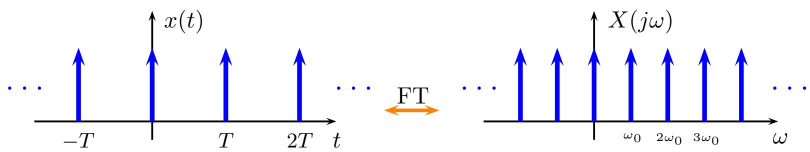
Example: $x(t) = \cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

$$X(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$



Example: Impulse Train

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} \delta(t-kT) \\ a_k &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_0 t}dt = \frac{1}{T} \quad \text{for all } k \\ X(j\omega) &= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \end{aligned}$$



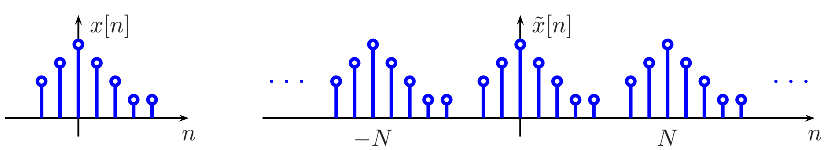
Discrete Time Fourier Transform (DTFT)

Chapter 5 *in* Oppenheim & Willsky

The discrete-time Fourier Transform (DTFT) is defined as:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}. \quad (5)$$

This definition is applicable to aperiodic signals and is motivated by arguments similar to those for the continuous-time Fourier Transform: for an aperiodic signal x of finite duration, construct periodic sequence $\tilde{x}$ of which x comprises one period, as shown below.



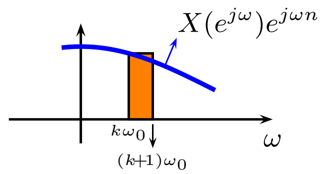
Then, from the analysis equation for Fourier Series,

$$\begin{aligned} a_k &= \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}[n]e^{-jk\frac{2\pi}{N}n} \\ Na_k &= \sum_{n=-\infty}^{\infty} x[n]e^{-jk\frac{2\pi}{N}n} = X(e^{j\omega})\Big|_{\omega=k\frac{2\pi}{N}} \end{aligned}$$

which means that $X(e^{j\omega})$ in (5) forms an envelope for the coefficients Na_k. As N increases, the fundamental frequency $\omega_0 = \frac{2\pi}{N}$ decreases and the harmonic components become closer in frequency, forming a continuum in the limit $N \to \infty$ that motivates the definition (5).

Similarly, the synthesis equation for Fourier Series gives

$$\begin{aligned}\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} &= \sum_{k=\langle N\rangle} \frac{1}{N} X(e^{jk\omega_0})e^{jk\omega_0 n} \\ &= \frac{1}{2\pi} \sum_{k=\langle N\rangle} \underbrace{\underbrace{\frac{2\pi}{N}}_{=\omega_0} X(e^{jk\omega_0})e^{jk\omega_0 n}}\end{aligned}$$



Thus, as $N \to \infty$, the summation recovers $\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$. To summarize:

$$\boxed{\begin{array}{rclr} x[n] &=& \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega & \text{(Synthesis Equation)} \\ X(e^{j\omega}) &=& \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} & \text{(Analysis Equation)} \end{array}} \quad (6)$$

Recall that in continuous time:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega, \qquad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

The main difference in discrete time is that $X(e^{j\omega})$ is periodic:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

and the synthesis equation in (6) requires integration over one period.

A result analogous to the theorem on page 1 guarantees convergence of the analysis equation in (6) when

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

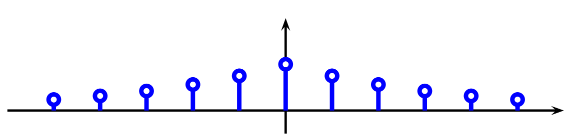
<u>Examples:</u>

1) For $x_1[n] = \delta[n]$, $X_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = 1$.

2) For $x_2[n] = a^n u[n]$, $|a| < 1$,

$$X_2(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

3) Now let $x[n] = a^{|n|}$, $|a| < 1$, which is depicted below:



Note that $x[n] = x_2[n] + x_2[-n] - x_1[n]$ where x_1 and x_2 are the signals in Examples 1 and 2, and the time reversal property (13) below states $x_2[-n] \longleftrightarrow X_2(e^{-j\omega})$. Thus, from linearity,

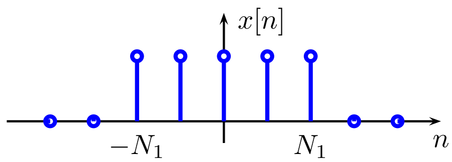
$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{1-ae^{-j\omega}} + \frac{1}{1-ae^{j\omega}} - 1 \\ &= \frac{2-a(e^{j\omega}+e^{-j\omega})}{1-a(e^{j\omega}+e^{-j\omega})+a^2} - 1 \\ &= \frac{2-2a\cos\omega}{1-2a\cos\omega+a^2} - 1 = \frac{1-a^2}{1+a^2-2a\cos\omega}. \end{aligned}$$

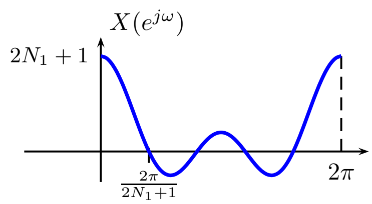
4)

$$x[n] = \begin{cases} 1 & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases}$$

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \begin{cases} \frac{\sin(\omega(N_1+1/2))}{\sin(\omega/2)} & \omega \neq 0, \\ 2N_1+1 & \omega = 0. \end{cases} \quad (7)$$

The derivation of (7) is similar to an example on page 2 of Lecture 3 notes.





Fourier Transform of Periodic Signals

Section 5.2 *in* Oppenheim & Willsky

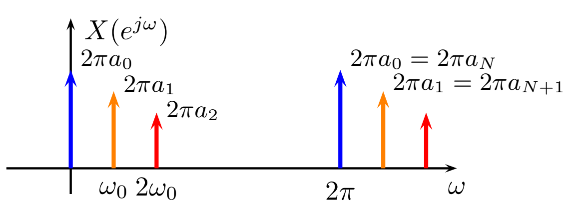
The following (generalized) Fourier Transform pairs are analogous to (1) and (3) in continuous time:

$$x[n] = 1 \quad \leftrightarrow \quad X(e^{j\omega}) = 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) \quad (8)$$

$$x[n] = e^{j\omega_0 n} \quad \leftrightarrow \quad X(e^{j\omega}) = 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l) \quad (9)$$

Using (9) and linearity, the Fourier Series of a periodic signal can be represented as a generalized Fourier Transform:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} \quad \leftrightarrow \quad X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi}{N}k\right) \quad (10)$$



Properties of DTFT

Section 5.3 *in* Oppenheim & Willsky

Time Shift:

$$\boxed{x[n-n_0] \longleftrightarrow e^{-j\omega n_0}X(e^{j\omega})} \tag{11}$$

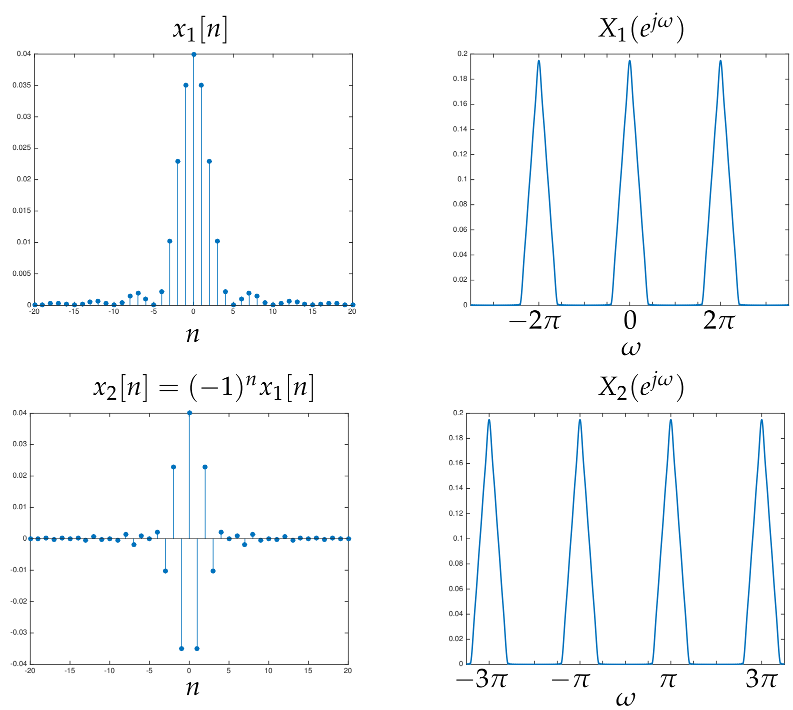
Frequency Shift:

$$\boxed{e^{j\omega_0 n}x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})} \tag{12}$$

As a special case let $\omega_0 = \pi$ and note that $e^{j\pi n} = (-1)^n$. Thus,

$$x_2[n] = (-1)^n x_1[n] \quad \Rightarrow \quad X_2(e^{j\omega}) = X_1(e^{j(\omega-\pi)}).$$

The figure below illustrates this with an example where $x_1[n]$ and $X_1(e^{j\omega})$ are shown at the top, and $x_2[n] = (-1)^n x_1[n]$ and $X_2(e^{j\omega})$ are at the bottom. Note that $X_1(e^{j\omega})$ is concentrated around $\omega = 0, \pm 2\pi, \cdots$ and $X_2(e^{j\omega})$ is concentrated around $\omega = \pm\pi, \pm 3\pi, \cdots$.



Example: Suppose a low-pass filter $H_{LP}(e^{j\omega})$ has been designed with impulse response $h_{LP}[n]$. To obtain a high-pass filter, let:

$$\begin{aligned} H_{HP}(e^{j\omega}) &= H_{LP}(e^{j(\omega-\pi)}) \\ h_{HP}[n] &= (-1)^n h_{LP}[n]. \end{aligned}$$

Time Reversal:

$$\boxed{x[-n] \longleftrightarrow X(e^{-j\omega})} \tag{13}$$