EE120 - Fall'19 - Lecture 5 Notes[1]

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12 September 2019

Fourier Series in Discrete-Time

Section 3.3 *in* Oppenheim & Willsky

Discrete-Time Periodic Signals

A discrete-time signal x is periodic if there exists integer $N \neq 0$ s.t.

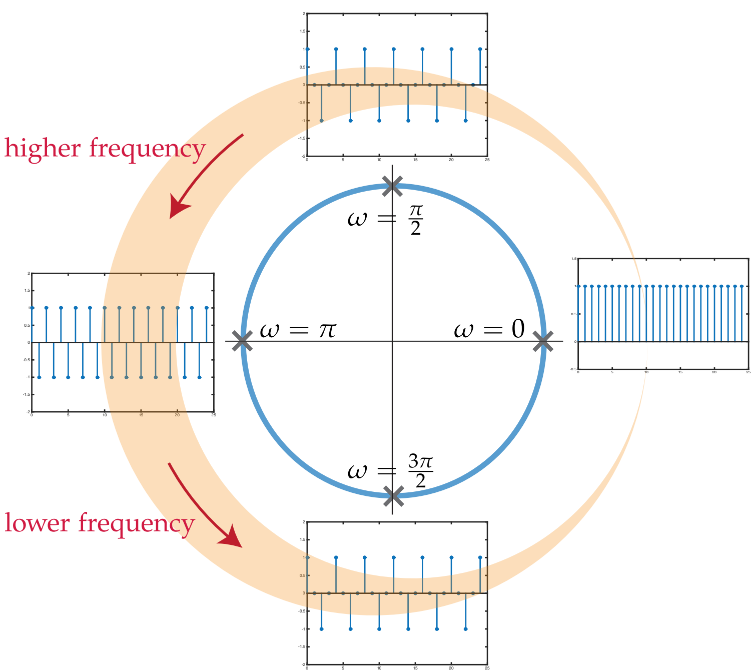
$$x[n+N] = x[n] \quad \text{for all } n.$$

Question: Is $x[n] = cos(\omega_0 n)$ periodic for any ω_0?
No, only when ω_0/π is rational. To find the fundamental period N, find the smallest integers M, N such that

$$\boxed{\omega_0 N = 2\pi M}$$

Examples:
1. $\cos(n)$ is not periodic;
2. $\cos(\frac{\pi}{5}n)$, $N = 10$;
3. $\cos(\frac{5\pi}{7}n)$, $N = 14$;
4. $\cos(\frac{5\pi}{7}n) + \cos(\frac{\pi}{5}n)$, $N = 70$, least common multiple of 14 and 10.

Recall that in discrete time $\omega = \pi$ is the highest frequency, as shown below:



Discrete-Time Fourier Series

Discrete-time Fourier Series expresses a sequence x with period N as a linear combination of:

$$\boxed{\Phi_k[n] := e^{jk\omega_0 n}, \quad k = 0, \mp 1, \mp 2, \ldots, \quad \omega_0 = \frac{2\pi}{N}.}$$

In continuous time each k defines a distinct function $e^{jk\omega_0 t}$. Here, by contrast, $e^{j(k+N)\omega_0 n} = e^{jk\omega_0 n}$ because $e^{jN\omega_0 n} = e^{j2\pi n} = 1$. Therefore,

$$\Phi_k[n] = \Phi_{k+N}[n] = \Phi_{k+2N}[n] = \ldots$$

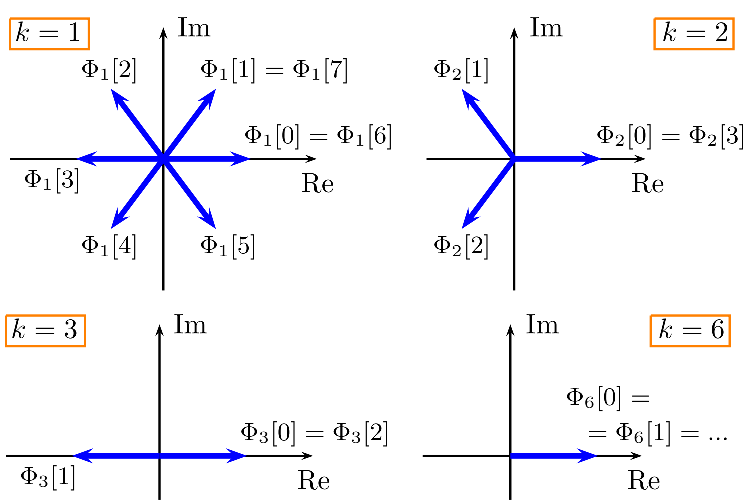
and N independent functions $\Phi_k[n]$ (*e.g.*, $\Phi_0[n], \Phi_1[n], \ldots, \Phi_{N-1}[n]$) are enough for discrete-time Fourier Series.

We thus use the finite sum:

$$\boxed{x[n] = \sum_{k=\langle N \rangle} a_k \Phi_k[n] \qquad \text{(Synthesis Equation)}} \tag{1}$$

where $k = \langle N \rangle$ means any set of N successive integers: $k = 0, 1, \ldots, N-1$, or $k = 1, 2, \ldots, N$, or other choices.

Example: For $N = 6$, $\Phi_k[n] = e^{jk\frac{2\pi}{6}n}$



Properties of $\Phi_k[n]$

1. Periodicity in n: $\Phi_k[n+N] = \Phi_k[n]$
2. Periodicity in k: $\Phi_{k+N}[n] = \Phi_k[n]$
3. Zero sum unless $k = 0 \,(\mathrm{mod} N)$:

$$\sum_{n=\langle N \rangle} \Phi_k[n] = \begin{cases} N & \text{if } k = 0, \mp N, \mp 2N, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

4. $\Phi_k[n]\Phi_m[n] = \Phi_{k+m}[n]$

Finding the Fourier Series coefficients a_k:

Multiplying both sides of (1) by $\Phi_{-m}[n]$ and summing over $n = \langle N \rangle$,

$$\begin{aligned}\sum_{n=\langle N\rangle} x[n]\Phi_{-m}[n] &= \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k\Phi_{k-m}[n] \\ &= \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle}\Phi_{k-m}[n] = Na_m \qquad (3)\end{aligned}$$

where the last equality follows because, from (2),

$$\sum_{n=\langle N\rangle}\Phi_{k-m}[n] = \begin{cases} N & \text{if } k = m \,(\text{mod} N) \\ 0 & \text{otherwise.}\end{cases}$$

Replacing the index m in (3) by k, we get:

$$\boxed{a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-j\frac{2\pi}{N}kn} \qquad \text{(Analysis Equation)}} \qquad (4)$$

As in continuous time, if x is real then $a_{-k} = a_k^*$. Combining with the periodicity of coefficients in discrete time ($a_{N-k} = a_{-k}$) we conclude:

$$\boxed{a_{N-k} = a_k^*.}$$

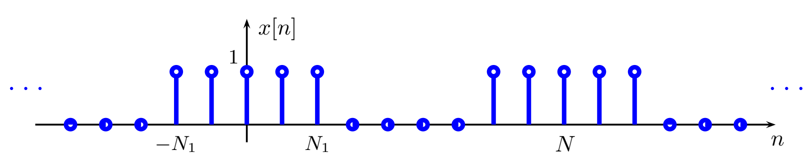
Example:

$$\begin{aligned} x[n] &= 1 + \underbrace{\sin\left(\frac{2\pi}{10}n\right)}_{=\frac{1}{2j}e^{j\frac{2\pi}{10}n} - \frac{1}{2j}e^{-j\frac{2\pi}{10}n}} + \underbrace{\cos\left(\frac{4\pi}{10}n + \frac{\pi}{4}\right)}_{=\frac{1}{2}e^{j\frac{\pi}{4}}e^{j\frac{4\pi}{10}n} + \frac{1}{2}e^{-j\frac{\pi}{4}}e^{-j\frac{4\pi}{10}n}} \qquad N = 10 \\ &= 1 + \frac{1}{2j}\Phi_1[n] - \frac{1}{2j}\Phi_{-1}[n] + \frac{1}{2}e^{j\frac{\pi}{4}}\Phi_2[n] + \frac{1}{2}e^{-j\frac{\pi}{4}}\Phi_{-2}[n]\end{aligned}$$

If we choose $\langle N \rangle$ to be $\{0, 1, 2, ..., 9\}$, then $a_3 = a_4 = a_5 = a_6 = a_7 = 0$,

$$a_0 = 1,\ a_1 = a_9^* = \frac{1}{2j},\ a_2 = a_8^* = \frac{1}{2}e^{j\frac{\pi}{4}}.$$

Example: Rectangular pulse train



For the special case $N_1 = 0$ ("impulse train"):

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}x[0]e^{-jk\frac{2\pi}{N}0} = \frac{1}{N} \quad \text{for all } k.$$

Derive the following for $N_1 \neq 0$:

$$a_k = \begin{cases} \frac{2N_1+1}{N} & k = 0 \\ \frac{1}{N}\frac{\sin(k\pi(2N_1+1)/N)}{\sin(k\pi/N)} & k \neq 0. \end{cases} \tag{5}$$

For $N = 9$, $N_1 = 2$, the figure below shows how the partial sum

$$\sum_{k=-M}^{M} a_k \Phi_k[n] \tag{6}$$

progressively reconstructs $x[n]$ as more harmonics are included.

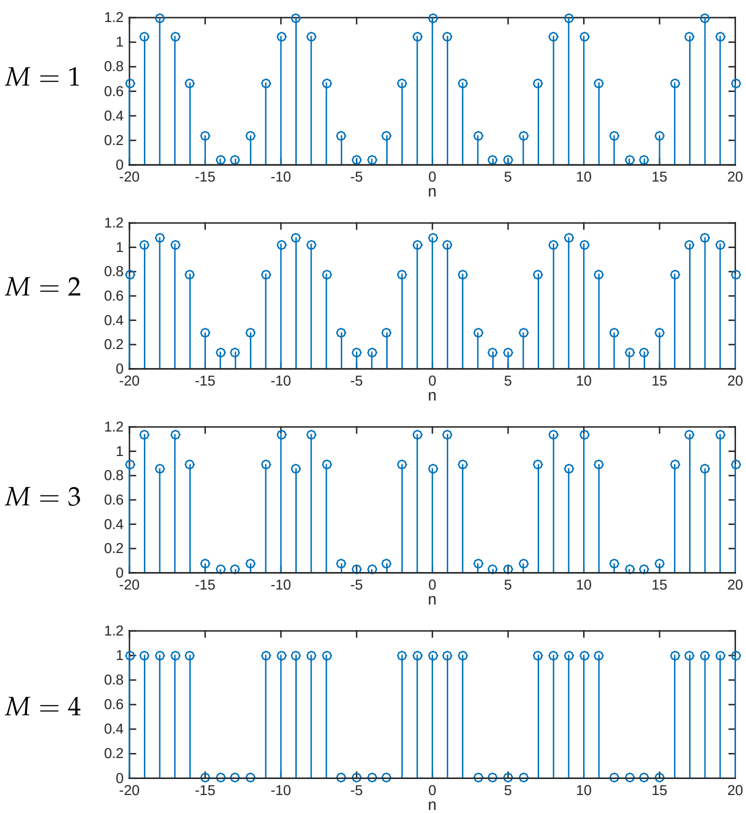


Figure 1: The partial sum (6) with Fourier coefficients (5), for $N = 9$ and $N_1 = 2$. When $M = 4$, (6) is the complete Fourier series; thus we fully recover the rectangular pulse.

Below is a summary of continuous- and discrete-time Fourier Series:

	Continuous Time	Discrete Time
Synthesis	$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}$
Analysis	$a_k = \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{T}t}$	$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\frac{2\pi}{N}n}$

Fourier Series as a Change of Basis

We can represent a discrete-time signal x with period N as a vector:

$$\vec{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

and define the inner product of two signals x and y as:

$$\vec{x} \cdot \vec{y} = \sum_{n=0}^{N-1} x[n] y[n]^* \tag{7}$$

which is the standard inner product in $\mathbb{C}^N$.

With this viewpoint we interpret the sequences $\Phi_k[n] = e^{jk\frac{2\pi}{N}n}$ as a set of basis vectors:

$$\vec{\Phi}_k = \begin{bmatrix} 1 \\ e^{jk\frac{2\pi}{N}} \\ \vdots \\ e^{jk\frac{2\pi}{N}(N-1)} \end{bmatrix}, \; k = 0, 1, \ldots, N-1,$$

and the Fourier Series representation (1) as a change of basis in $\mathbb{C}^N$:

$$\vec{x} = a_0 \vec{\Phi}_0 + a_1 \vec{\Phi}_1 + \cdots + a_{N-1} \vec{\Phi}_{N-1}. \tag{8}$$

In fact these basis vectors are orthogonal to each other, because

$$\begin{aligned} \vec{\Phi}_k \cdot \vec{\Phi}_m &= \sum_{n=0}^{N-1} \Phi_k[n] \Phi_m[n]^* = \sum_{n=0}^{N-1} e^{jk\frac{2\pi}{N}n} e^{-jm\frac{2\pi}{N}n} = \sum_{n=0}^{N-1} \Phi_{k-m}[n] \\ &= \begin{cases} N & \text{if } k = m \, (\text{mod} N) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

from (2). Orthogonality simplifies the computation of the coefficients in (8). If we take the inner product of both sides of (8) with $\vec{\Phi}_k$, then

$$\vec{x} \cdot \vec{\Phi}_k = a_k (\vec{\Phi}_k \cdot \vec{\Phi}_k) = a_k N.$$

Therefore,

$$a_k = \frac{1}{N} \vec{x} \cdot \vec{\Phi}_k, \tag{9}$$

which yields the analysis equation (4) when expanded using the inner product definition (7). Thus the kth term in the Fourier Series is essentially the projection of the signal onto the kth basis vector $\vec{\Phi}_k$.

The advantage of the Fourier basis is that, instead of the values in time $x[0], x[1], \ldots, x[N-1]$, it represents the signal with the coefficients $a_0, a_1, \ldots, a_{N-1}$ describing its frequency content.