EE120 - Fall'19 - Lecture 10 Notes[1]

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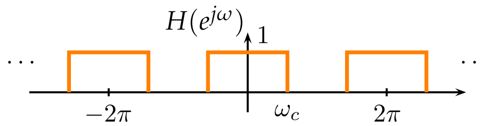
1 October 2019

FIR Filter Design by Windowing

Ideal Low-Pass Filter:

$$h[n] = \frac{\omega_c}{\pi}\text{sinc}\left(\frac{\omega_c}{\pi}n\right) \longleftrightarrow$$



To obtain a FIR filter truncate the ideal impulse response:

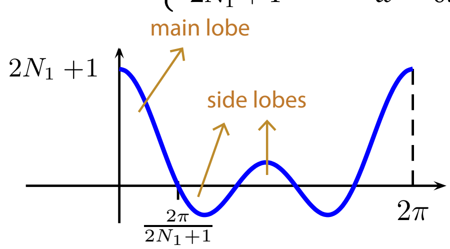
$$\hat{h}[n] = h[n]w[n], \quad \text{where} \quad w[n] = \begin{cases} 1 & |n| \leq N_1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the effect of truncation on the frequency response? From the multiplication property of DTFT (Lecture 9):

$$\hat{H}(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} H(e^{j\theta})W(e^{j(\omega-\theta)})d\theta \tag{1}$$

and, from Lecture 8 (page 5), the DTFT of the rectangular window w above is:

$$W(e^{j\omega}) = \begin{cases} \frac{\sin(\omega(N_1+1/2))}{\sin(\omega/2)} & \omega \neq 0, \\ 2N_1+1 & \omega = 0. \end{cases}$$



Gibbs Phenomenon and Tapered Windows

The periodic convolution (1) is depicted in the Figure 1 below. Figure 2 shows that $\hat{H}(e^{j\omega})$ exhibits oscillations near the discontinuities of $H(e^{j\omega})$ and their amplitudes do not decrease as N_1 is increased. This is known as the Gibbs Phenomenon.

These oscillations are caused by the sizable side lobes of $W(e^{j\omega})$ (high frequency components) which are due to the abrupt change from 0 to 1 in the rectangular window function w above.

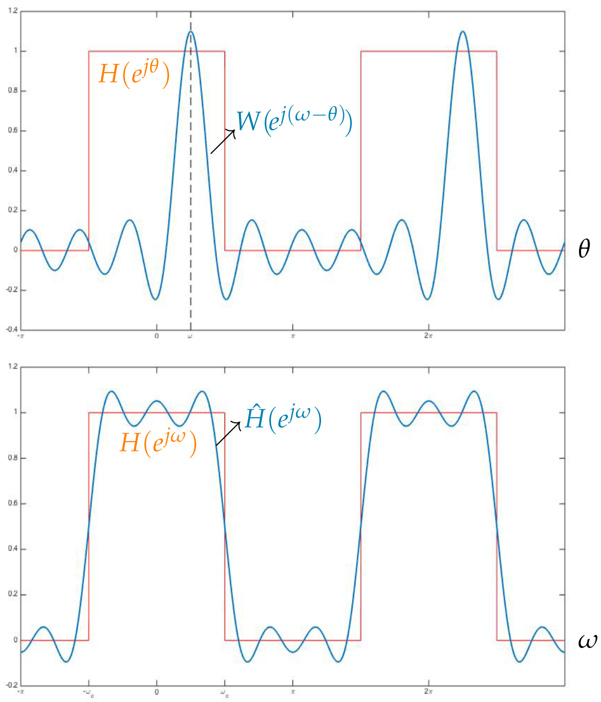


Figure 1: The periodic convolution (1) of the ideal low-pass filter response and the DTFT of the rectangular window.

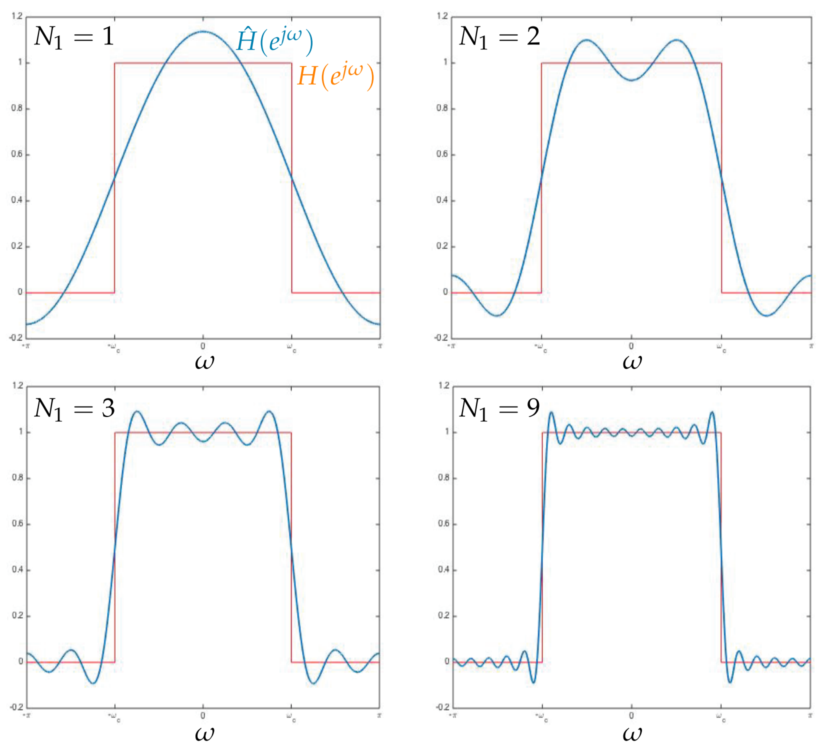


Figure 2: The effect of increasing the window length. $\hat{H}(e^{j\omega})$ exhibits oscillations near the discontinuities of $H(e^{j\omega})$ and their amplitudes do not decrease as N_1 is increased. This is known as the Gibbs Phenomenon.

"Tapered" windows mitigate this problem, e.g., the triangular (Bartlett) window:

$$w[n] = \begin{cases} 1 - \frac{|n|}{N_1} & \text{if } |n| \leq N_1 \\ 0 & \text{otherwise.} \end{cases}$$

Other tapered windows exist (Hanning, Hamming, Blackman, etc.) and are depicted in Figure 3. Although the differences between these windows may not be appreciable in time domain, their Fourier Transforms have significant differences as shown in Figure 4. Note the tradeoff between main lobe width & side lobe amplitude in Figure 4.

must be small for sharp transition from passband to stopband

must be small to reduce ripples

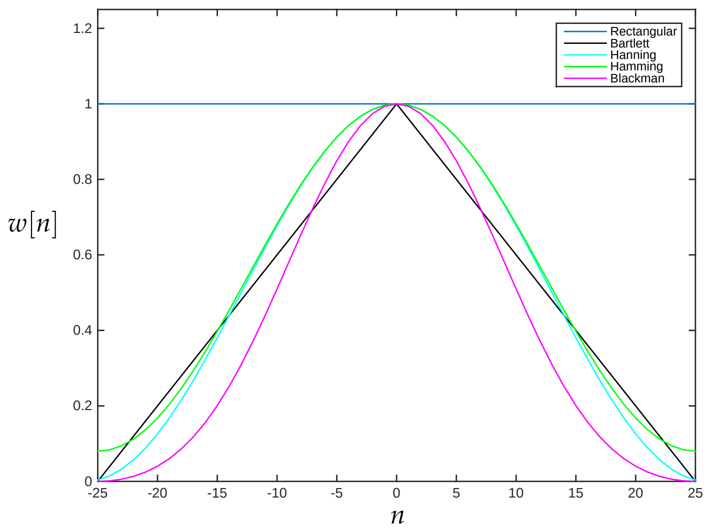


Figure 3: Tapered windows of type Bartlett, Hanning, Hamming, and Blackman for $N_1 = 25$ superimposed.

Summary: To obtain a FIR filter truncate the ideal filter's impulse response $h[n]$ with one of the window functions $w[n]$:

$$\hat{h}[n] = h[n]w[n].$$

The new impulse response is zero outside of $n \in \{-N_1, \cdots, N_1\}$ but not yet causal. To make it causal, shift to the right by N_1:

$$\hat{h}[n - N_1] \longleftrightarrow e^{-j\omega N_1}\hat{H}(e^{j\omega})$$

which does not change the magnitude of the frequency response, only the phase. Finally, $\hat{h}[n]$ must be scaled by a constant to obtain $\sum_n \hat{h}[n] = 1$, so the dc gain is $\hat{H}(e^{j0}) = 1$.

FIR implementation:

$$y[n] = b_0 x[n] + b_1 x[n-1] + ... + b_M x[n-M]$$

where $b_0, ..., b_M$ are the impulse response coefficients: $b_n = \hat{h}[n - N_1]$.

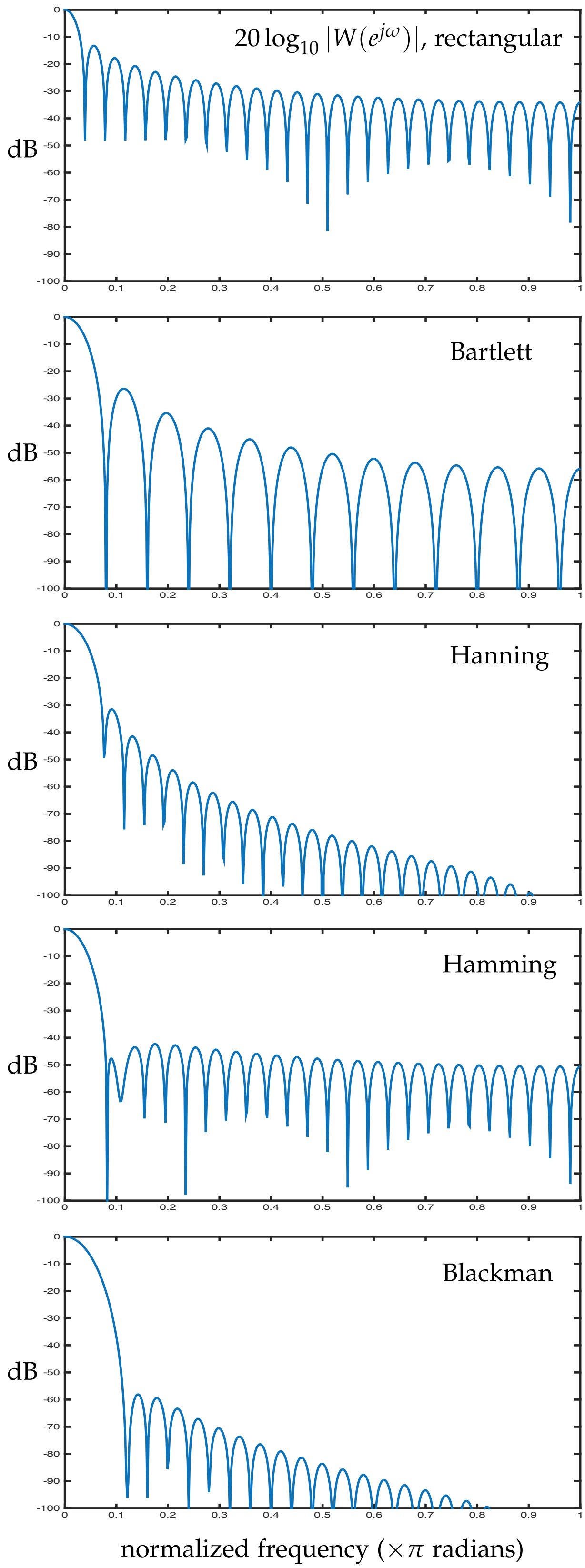


Figure 4: The magnitude of the Fourier Transform $W(e^{j\omega})$ (in dB) for the rectangular, Bartlett, Hanning, Hamming, and Blackman windows. Here each window function in Figure 3 is scaled such that $\sum_n w[n] = 1$, so $W(e^{j0}) = 1 = 0$ dB. Note that the tapered windows progressively reduce the side lobe amplitudes in the order they are presented. This has the desired effect of reducing ripples in the frequency response of the truncated filter. However, the main lobe width increases which has a negative effect: the transition from passband to stopband will be slower for the filter truncated with the respective window.

Discrete Fourier Transform (DFT)

The DFT of a sequence $x[n]$ with finite length, say N, is a sequence $X[k]$ of the same length in the frequency domain:

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} \quad k = 0, 1, \ldots, N-1} \tag{2}$$

As a convention we represent the length-N sequences $x[n]$ and $X[k]$ with indices n and k running from 0 to $N-1$.

Connection to Discrete Fourier Series

If we define a period-N sequence $\tilde{x}[n]$ by adding $x[n]$ end-to-end:

$$\tilde{x}[n] := x[(n \bmod N)] \tag{3}$$

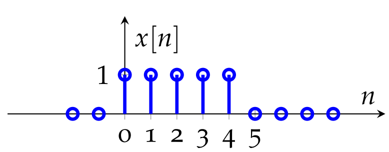
then $X[k] = Na_k$ for $k = 0, 1, \ldots, N-1$:

$$Na_k := X[(k \bmod N)]. \tag{4}$$

Synthesis Equation for DFT:

$$\boxed{x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn} \quad n = 0, 1, \ldots, N-1} \tag{5}$$

Example:

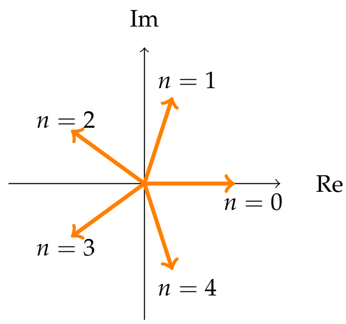


Take $N = 5$ (5-point DFT):

$$X[k] = \sum_{n=0}^{4} e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, 2, 3, 4 \tag{6}$$

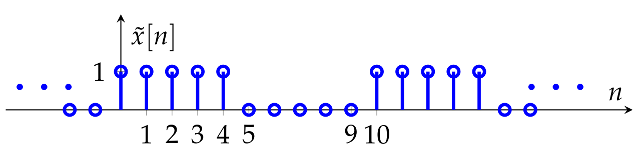
$$= \begin{cases} 5 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

To see why $X[k] = 0$ for, say $k = 1$, note that (6) is the sum of the following complex numbers:



What if we take $N = 10$ (10-point DFT)?

$X[k] = 10a_k, k = 0, 1, \ldots, 9$ where a_k are FS coefficients of:



This is a rectangular pulse train similar to the one studied in Lecture 5 with $N_1 = 2, N = 10$, except that the pulse is not centered at $n = 0$, but at $n = 2$. To account for this time shift we multiply the kth Fourier Series coefficient in Lecture 5 with $e^{-j\frac{4\pi}{10}k}$ (explain why). Then,

$$X[0] = Na_0 = N\frac{2N_1 + 1}{N} = 2N_1 + 1 = 5$$
$$X[k] = Na_k = Ne^{-j\frac{4\pi}{10}k}\frac{1}{N}\frac{\sin(k\pi(2N_1+1)/N)}{\sin(k\pi/N)} = e^{-j\frac{4\pi}{10}k}\frac{\sin\left(\frac{\pi}{2}k\right)}{\sin\left(\frac{\pi}{10}k\right)}$$

for $k = 1, \ldots, 9$. From the expression above we compute:

$X[0] = 5, X[2] = X[4] = X[6] = X[8] = 0, X[1] = 1 - j3.0777,$ $X[3] = 1 - j0.7265, X[5] = 1, X[7] = 1 + j0.7265, X[9] = 1 + j3.0777.$

Conjugate Symmetry Property of the DFT:

If $x[n]$ is real, then

$$\boxed{X^*[N-k] = X[k] \quad k = 1, 2, \ldots, N-1.} \tag{8}$$

This follows from the analogous property of Fourier Series coefficients.

Example: In the 10-point DFT above, $X[1] = X^*[9]$, $X[3] = X^*[7], \ldots$.

Connection between DFT and DTFT

Recall:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} \tag{9}$$

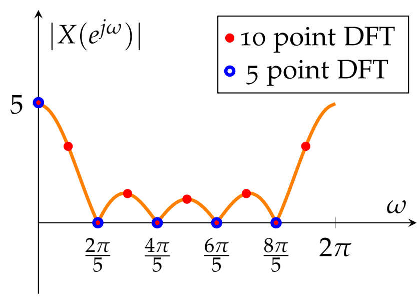
$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, \dots, N-1 \tag{10}$$

$$\boxed{X[k] = X(e^{j\omega})\Big|_{w=\frac{2\pi}{N}k}} \qquad \begin{array}{r} N \text{ samples of DTFT at } \omega = \frac{2\pi}{N}k \\ k = 0, 1, \dots, N-1. \end{array} \tag{11}$$

Note that (11) is valid only if the duration of x is $\leq N$, because in (9) we assumed $x[n] = 0$ when $n \notin \{0, 1, \dots, N-1\}$.

Back to the example:

$$X(e^{j\omega}) = \sum_{n=0}^{4} e^{-j\omega n} = \begin{cases} 5 & \omega = 0 \\ e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)} & \omega \neq 0 \end{cases} \tag{12}$$



DFT Makes Convolution Easy

$$y[n] = \underbrace{h[n]}_{\substack{\text{FIR with} \\ \text{duration} = P}} * \underbrace{x[n]}_{\substack{\text{input sequence} \\ \text{with duration} = L}} \tag{13}$$

Set $N \geq L + P - 1$ (duration of y). Pad zeros in h and x to make them duration $= N$. Take their N-point DFT to find $H[k]$ and $X[k]$. Take inverse DFT of $H[k]X[k]$ to obtain $y[n]$.