EE120 - Fall'19 - Lecture 7 Notes[1]

Murat Arcak

19 September 2019

[1] Licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.

Continuous Time Fourier Transform Continued

The Convolution Property of the Fourier Transform

Section 4.4 *in* Oppenheim & Willsky

$$\boxed{(x_1 * x_2)(t) \overset{FT}{\longleftrightarrow} X_1(\omega)X_2(\omega)} \tag{1}$$

Proof: Recall $(x_1 * x_2)(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau$ and apply the Fourier Transform:

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} x_1(\tau)\underbrace{\int_{-\infty}^{\infty} x_2(t-\tau)e^{-j\omega t}dtd\tau}_{=e^{-j\omega\tau}X_2(\omega)} \\
&= \left(\int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}d\tau\right)X_2(\omega) \\
&= X_1(\omega)X_2(\omega).
\end{aligned}$$

Note that we used the time-shift property $x_2(t-\tau) \overset{FT}{\leftrightarrow} e^{-j\omega\tau}X_2(\omega)$ in the second line.

<u>Example 1</u>: Use the convolution property to compute $(x_1 * x_2)(t)$ where $x_1(t) = e^{-at}u(t)$, $x_2(t) = e^{-bt}u(t)$, $a, b > 0$ and $a \neq b$.

$$\begin{aligned}
x_1(t) = e^{-at}u(t),\ a > 0 \quad &\leftrightarrow \quad X_1(\omega) = \frac{1}{a+j\omega} \\
x_2(t) = e^{-bt}u(t),\ b > 0 \quad &\leftrightarrow \quad X_2(\omega) = \frac{1}{b+j\omega}
\end{aligned}$$

$$X_1(\omega)X_2(\omega) = \frac{1}{(a+j\omega)(b+j\omega)} \tag{2}$$

Now, with a *partial fraction expansion*, we express this product as

$$\frac{A}{a+j\omega} + \frac{B}{b+j\omega} = \frac{(Ab+Ba)+j(A+B)\omega}{(a+j\omega)(b+j\omega)} \tag{3}$$

which matches (2) if $Ab + Ba = 1$ and $A + B = 0$, that is $A = \frac{1}{b-a}$ and $B = -\frac{1}{b-a}$. Thus,

$$X_1(\omega)X_2(\omega) = \frac{A}{a+j\omega} + \frac{B}{b+j\omega} = \frac{1}{b-a}\frac{1}{a+j\omega} - \frac{1}{b-a}\frac{1}{b+j\omega}$$

where the first term and second terms are the Fourier Transforms of $\frac{1}{b-a}e^{-at}u(t)$ and $-\frac{1}{b-a}e^{-bt}u(t)$, respectively. Therefore,

$$(x_1 * x_2)(t) = \frac{1}{b-a}\left(e^{-at} - e^{-bt}\right)u(t).$$

Note how we simplified the inverse Fourier Transform computation for (2) with the partial fraction expansion (3). Instead of directly applying the synthesis equation to (2) and evaluating a cumbersome integral, we broke down the product (2) into a sum of simpler terms whose inverse Fourier Transforms were apparent.

Derivative Properties of the Fourier Transform

The first property below follows by taking the derivative of both sides of the synthesis equation. The second property follows by taking the derivative of both sides of the analysis equation.

$$\boxed{\frac{dx(t)}{dt} \overset{FT}{\leftrightarrow} j\omega X(\omega)} \tag{4}$$

$$\boxed{-jtx(t) \overset{FT}{\leftrightarrow} \frac{dX(\omega)}{d\omega}} \tag{5}$$

Note the duality in (4)-(5): derivative in time domain involves multiplication with ω in frequency domain, and derivative in frequency domain involves multiplication with t in time domain.

Repeated applications of (4) and (5) result in the properties:

$$\frac{d^k x(t)}{dt^k} \overset{FT}{\leftrightarrow} (j\omega)^k X(\omega) \quad \text{and} \quad (-jt)^k x(t) \overset{FT}{\leftrightarrow} \frac{d^k X(\omega)}{d\omega^k}.$$

<u>Example 2</u>: Compute $(x * x)(t)$ where $x(t) = e^{-at}u(t)$, $a > 0$, which is the case $a = b$ disallowed in Example 1 above. This time we need to find the inverse Fourier Transform of

$$\frac{1}{(a+j\omega)^2}$$

which can no longer be expanded as in (3). (Why not?) Instead note

$$\frac{1}{(a+j\omega)^2} = j\frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right) = j\frac{dX(\omega)}{d\omega}$$

which, according to (5), is the Fourier Transform of

$$-j^2 tx(t) = tx(t) = te^{-at}u(t).$$

Thus, $(x * x)(t) = te^{-at}u(t)$. You can extend this example to show

$$\underbrace{(x * \ldots * x)}_{\text{r convolutions}}(t) = \frac{t^r}{r!}e^{-at}u(t).$$

The Frequency Shifting Property

Section 4.3 *in* Oppenheim & Willsky

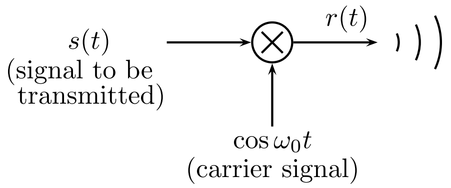
$$\boxed{e^{j\omega_0 t}x(t) \overset{FT}{\longleftrightarrow} X(\omega-\omega_0)} \tag{6}$$

dual of $x(t-t_0) \leftrightarrow e^{-j\omega_0 t}X(j\omega)$

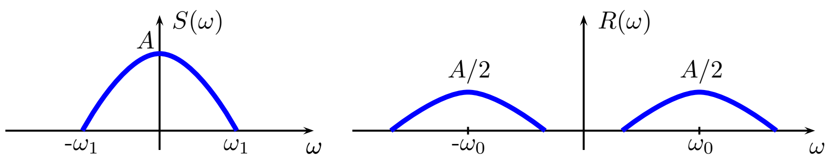
Proof:

$$\int_{-\infty}^{\infty} e^{j\omega_0 t}x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt = X(\omega-\omega_0)$$

Example: Amplitude Modulation (AM)

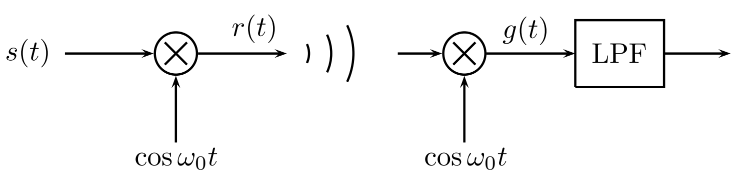


$$r(t) = s(t)\left(\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right) \longleftrightarrow R(\omega) = \frac{1}{2}S(\omega-\omega_0) + \frac{1}{2}S(\omega+\omega_0)$$

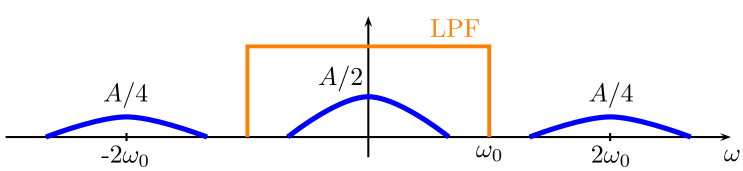


($\omega_0 >> \omega_1$ in practice)

Demodulation:



$$G(\omega) = \frac{1}{2}R(\omega-\omega_0) + \frac{1}{2}R(\omega+\omega_0)$$



If the low-pass filter (LPF) has a gain of 2 in the passband ($|\omega| \leq \omega_0$) then we recover the signal $s(t)$ exactly.

The Multiplication Property

Section 4.5 *in* Oppenheim & Willsky

$$\boxed{s(t)p(t) \longleftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\theta)P(\omega-\theta)d\theta} \tag{7}$$

dual of the convolution property

Proof: Apply the synthesis equation to the right-hand side above:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\theta)P(\omega-\theta)d\theta\right)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\theta)\underbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} P(\omega-\theta)e^{j\omega t}d\omega\right)}_{=e^{j\theta t}p(t)}d\theta \\
&= p(t)\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\theta)e^{j\theta t}d\theta}_{=s(t)} = p(t)s(t).
\end{aligned}$$

We used the frequency shift property in the second line and the synthesis equation in the third line.

Back to LTI Systems

Recall from Lecture 3:

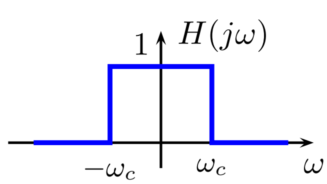
$$x(t) = e^{j\omega t} \rightarrow \boxed{h(t)} \rightarrow y(t) = H(j\omega)e^{j\omega t} \tag{8}$$

where

$$\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt \quad \text{"frequency response"}} \tag{9}$$

Note the frequency response is nothing but the Fourier Transform of the impulse response $h(t)$. We henceforth use the notation $H(j\omega)$ for Fourier Transforms instead of $H(\omega)$, where the presence of j distinguishes the frequency response from the transfer function (Lecture 3). This notation will also distinguish the continuous-time Fourier Transform from the discrete-time Fourier Transform studied later.

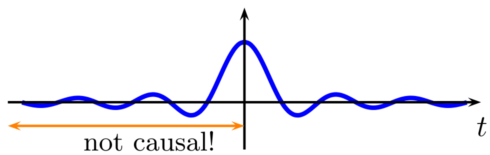
Example: An ideal low pass filter with cutoff frequency ω_c has the frequency response:



From Lecture 6 the inverse Fourier Transform, that is the impulse response, is

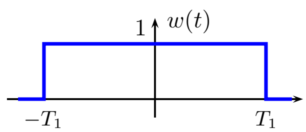
$$h(t) = \frac{\omega_c}{\pi}\text{sinc}\left(\frac{\omega_c}{\pi}t\right) \tag{10}$$

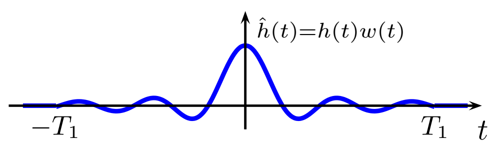
which fails the causality test ($h(t) = 0$ for all $t < 0$).



This is why the ideal low pass filter is "ideal" and not appropriate for a causal implementation. To satisfy the causality constraint we will truncate and shift the impulse response so that $h(t) = 0$ when $t < 0$, and explore how much the frequency response degrades as a result.

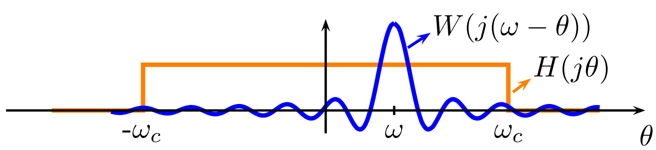
To truncate the impulse response we multiply it with the window function $w(t)$ shown below, so that the product is zero when $|t| > T_1$:



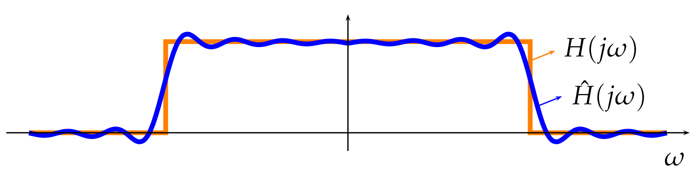


Then, from the multiplication property,

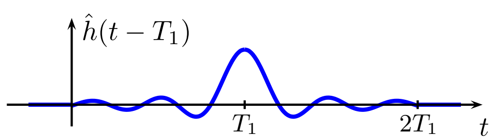
$$\hat{h}(t) := h(t)w(t) \quad \overset{FT}{\leftrightarrow} \quad \hat{H}(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\theta)W(j(\omega-\theta))d\theta$$



Since the Fourier Transform $W(j\omega)$ of the window function is a sinc, convolution results in the frequency response $\hat{H}(j\omega)$ below:



Next we shift $\hat{h}$ to the the right by T_1 to achieve causality:



From the time-shift property of Fourier Transforms

$$\hat{h}(t-T_1) \quad \overset{FT}{\leftrightarrow} \quad e^{-j\omega T_1}\hat{H}(j\omega)$$

Since $\left|e^{-j\omega T_1}\hat{H}(j\omega)\right| = |\hat{H}(j\omega)|$, the frequency response of the resulting causal filter has the same magnitude as $\hat{H}(j\omega)$ above, but it has an additional phase of $-\omega T_1$ due to the time shift.

Frequency Response of Continuous Time LTI Systems

Sections 4.7 and 6.5 *in* Oppenheim & Willsky

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{11}$$

Take Fourier transforms of both sides and apply the differentiation property:

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega) \tag{12}$$

Since $Y(j\omega) = H(j\omega)X(j\omega)$ by the convolution property,

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}. \tag{13}$$

Example: First-order system:

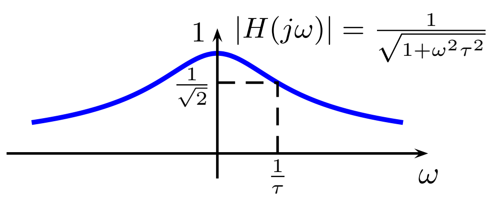
$$\boxed{\tau \frac{dy}{dt} + y(t) = x(t)} \tag{14}$$

where $\tau > 0$ is often called the "time constant." By the formula (13), the frequency response is:

$$H(j\omega) = \frac{1}{1 + j\omega\tau} \tag{15}$$

and its magnitude is as shown below. From the inverse Fourier Transform of (15), we conclude that the impulse response is:

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t).$$



Example: Second-order system:

$$\boxed{\frac{d^2 y}{dt^2} + 2\xi\omega_n \frac{dy}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)} \tag{16}$$

where ξ is called the damping ratio, and ω_n the natural frequency.

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2} = \frac{1}{(j\omega/\omega_n)^2 + 2\xi(j\omega/\omega_n) + 1}$$

The figure below shows the frequency, impulse, and step responses for various values of ξ. Note that increasing ω_n stretches the frequency response along the ω axis and compresses the impulse and step responses along the t axis. Therefore, a large natural frequency means faster response.

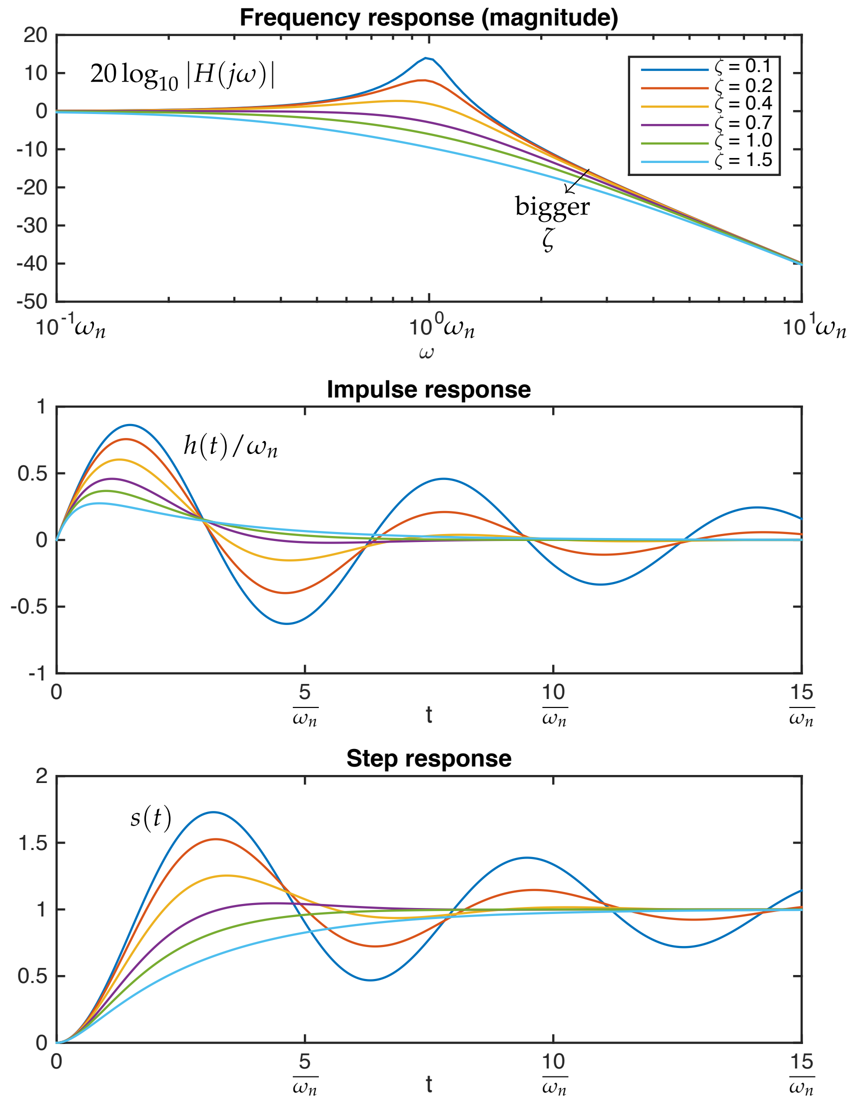


Figure 1: The frequency, impulse, and step responses for the second order system (16). Note from the frequency response (top) that a resonance peak occurs when $\zeta < 0.7$.

When does resonance occur?

$$|H(j\omega)|^2 = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2 \frac{\omega^2}{\omega_n^2}} = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^4 + (4\xi^2 - 2)\left(\frac{\omega}{\omega_n}\right)^2 + 1}$$

Note that the denominator is strictly increasing in ω if $4\xi^2 - 2 \geq 0$ and has a minimum at some $\omega > 0$ otherwise. Thus, if $4\xi^2 - 2 < 0$ (i.e., $\xi < 1/\sqrt{2} \approx 0.7$), then $|H(j\omega)|$ has a resonance peak as confirmed with the frequency response in the top figure above.