EE120 - Fall'19 - Lecture 6 Notes[1]

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Continuous Time Fourier Transform

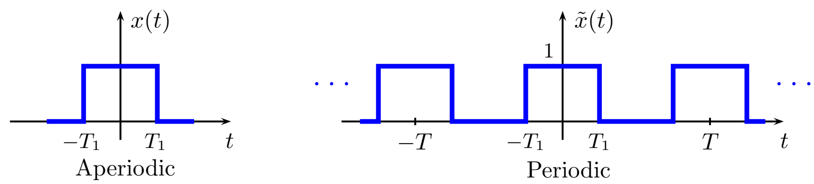
Chapter 4 *in* Oppenheim & Willsky

Unlike Fourier Series, the Fourier Transform is applicable to *aperiodic* signals. It has the form

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{1}$$

where ω is a continuous frequency variable. To motivate this definition we treat the aperiodic signal x as the limit of a periodic signal $\tilde{x}$ as period $T \to \infty$ (see example below). As T increases, the fundamental frequency $\omega_0 = \frac{2\pi}{T}$ decreases and the harmonic components become closer in frequency, forming a continuum in the limit $T \to \infty$.

Example 1:



The definition (1) applied to the aperiodic signal x gives

$$X(\omega) = \int_{-T_1}^{T_1} e^{-j\omega t} dt = \begin{cases} 2T_1 & \omega = 0 \\ \frac{1}{-j\omega} e^{-j\omega t}\Big|_{-T_1}^{T_1} = \frac{2 sin(\omega T_1)}{\omega} & \omega \neq 0. \end{cases} \tag{2}$$

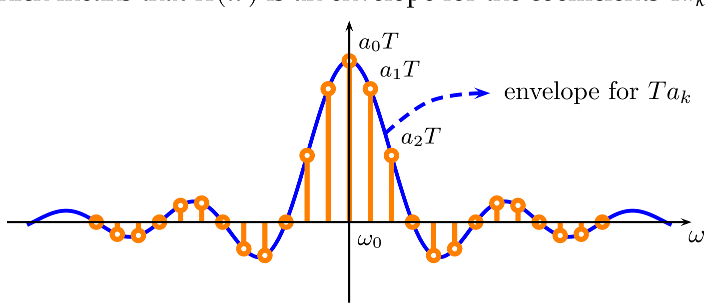
Now recall from Lecture 4 that $\tilde{x}$ has Fourier Series coefficients:

$$a_k = \begin{cases} \frac{2T_1}{T} & k = 0 \\ \frac{2 sin(k\omega_0 T_1)}{k\omega_0 T} & k \neq 0 \end{cases} \tag{3}$$

where $\omega_0 = \frac{2\pi}{T}$. Comparing (2) and (3), we see that

$$Ta_k = X(\omega)|_{\omega = k\omega_0} \tag{4}$$

which means that $X(\omega)$ is an envelope for the coefficients Ta_k:



Thus, the Fourier Transform of x emerges from the Fourier Series coefficients of $\tilde{x}$, which get densely packed as $T \to \infty$ and form the silhouette of a function, $X(\omega)$, of a continuous frequency variable ω.

The square pulse example above is easy to generalize to any function x of finite duration. Create periodic signal $\tilde{x}$ as above, with T large enough to avoid overlaps. Then,

$$a_k = \frac{1}{T}\int_T \tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt$$

if we integrate over an interval encompassing the full duration of x. It follows that

$$Ta_k = \int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt = X(\omega)|_{\omega=k\omega_0} \quad (5)$$

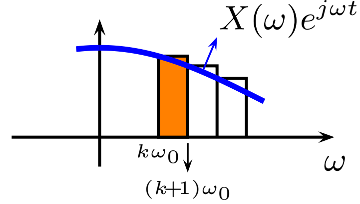
where the envelope $X(\omega)$ is as defined in (1).

To reconstruct $x(t)$ from its Fourier Transform $X(\omega)$, recall from the synthesis equation for Fourier Series that

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

and substitute a_k from (5):

$$\begin{aligned}\tilde{x}(t) = \sum_{k=-\infty}^{\infty}\left(\frac{1}{T}X(\omega)\right)\Big|_{\omega=k\omega_0} e^{jk\omega_0 t} &= \sum_{k=-\infty}^{\infty}\frac{1}{T}\left(X(\omega)e^{j\omega t}\right)\Big|_{\omega=k\omega_0} \\ &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\omega_0\Big(X(\omega)e^{j\omega t}\Big)\Big|_{\omega=k\omega_0}\end{aligned}$$



The kth term in this summation can be pictured as the shaded bar in the figure on the right. Thus, as $T \to \infty$ ($\omega_0 \to 0$), the summation converges to the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega. \quad (6)$$

Since $\tilde{x}$ recovers x in the limit as $T \to \infty$, this expression serves as the synthesis equation to reconstruct $x(t)$. To summarize:

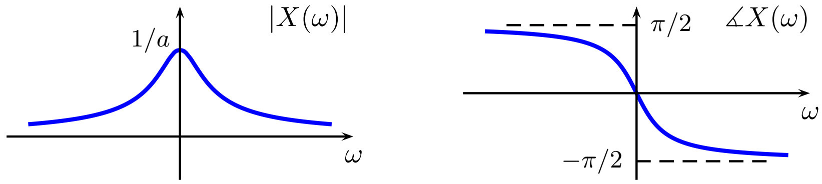
$X(\omega)$	$=$	$\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$	(Analysis Equation)
$x(t)$	$=$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	(Synthesis Equation)

Example 2: For $x(t) = e^{-at}u(t),\ a > 0$,

$$X(\omega) = \int_0^{\infty} e^{-at}e^{-j\omega t}dt = \int_0^{\infty} e^{-(a+j\omega)t}dt = \frac{-1}{a+j\omega}\underbrace{e^{-(a+j\omega)t}\Big|_0^{\infty}}_{=-1}$$

$$X(\omega) = \frac{1}{a+j\omega}, \quad |a+j\omega| = \sqrt{a^2+\omega^2}, \quad \measuredangle(a+j\omega) = tan^{-1}(\omega/a)$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \quad \measuredangle X(\omega) = -\tan^{-1}(\omega/a)$$



Example 3: Given the Fourier Transform

$$X(\omega)=\begin{cases}1 & |\omega|<\pi\\ 0 & |\omega|\geq\pi\end{cases} \tag{7}$$

find

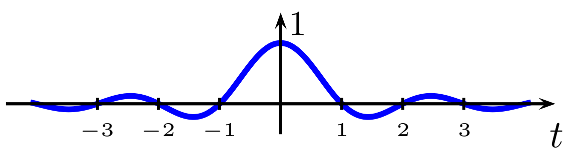
$$x(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{j\omega t}d\omega.$$

When $t=0$ the integral gives $x(0)=1$. When $t\neq 0$,

$$x(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{j\omega t}d\omega=\frac{1}{2\pi}\frac{1}{jt}e^{j\omega t}\Big|_{-\pi}^{\pi}=\frac{e^{j\pi t}-e^{-j\pi t}}{2j\pi t}=\frac{\sin\pi t}{\pi t}$$

Thus,

$$x(t)=\mathrm{sinc}(t):=\begin{cases}1 & t=0\\ \frac{\sin\pi t}{\pi t} & t\neq 0.\end{cases}$$



The Fourier Transform (2) in Example 1 can be expressed as a (scaled) sinc function as well:

$$X(\omega)=2T_1\mathrm{sinc}\left(\frac{T_1}{\pi}\omega\right). \tag{8}$$

Note the duality in Examples 1 and 3:

$$\boxed{\begin{array}{rcl}\text{rectangular pulse} & \overset{FT}{\leftrightarrow} & \text{sinc}\\ \text{sinc} & \overset{FT}{\leftrightarrow} & \text{rectangular pulse}\end{array}}$$

Properties of the Fourier Transform

Section 4.3 *in* Oppenheim & Willsky

Consider two signals $x(t)\overset{FT}{\leftrightarrow}X(\omega)$ and $y(t)\overset{FT}{\leftrightarrow}Y(\omega)$.

Linearity: For any constants a, b,

$$\boxed{ax(t)+by(t)\overset{FT}{\leftrightarrow}aX(\omega)+bY(\omega)} \tag{9}$$

Time-Shift:

$$\boxed{x(t-t_0) \overset{FT}{\leftrightarrow} e^{-j\omega t_0} X(\omega)} \tag{10}$$

Proof:
$$\begin{aligned}\int_{-\infty}^{\infty} x(\underbrace{t-t_0}_{\triangleq\tau})e^{-j\omega t}dt &= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(t_0+\tau)}d\tau \\ &= e^{-j\omega t_0}\underbrace{\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau}_{=X(\omega)}\end{aligned}$$

Conjugation and Conjugate Symmetry:

$$\boxed{x^*(t) \overset{FT}{\leftrightarrow} X^*(-\omega)} \tag{11}$$

If $x(t)$ is real: $X(\omega) = X^*(-\omega)$ (because $x(t) = x^*(t)$)

$$\Rightarrow \quad |X(\omega)| = |X(-\omega)| \quad \text{(even symmetry)} \tag{12}$$
$$\measuredangle X(\omega) = -\measuredangle X(-\omega) \quad \text{(odd symmetry)} \tag{13}$$

You can see such symmetry in the plots of Example 2 above.

Differentiation:

$$\boxed{\frac{dx(t)}{dt} \overset{FT}{\leftrightarrow} j\omega X(\omega)} \tag{14}$$

Proof: Take the derivative of both sides of the synthesis equation.

Time and Frequency Scaling:

$$\boxed{x(at) \overset{FT}{\leftrightarrow} \frac{1}{|a|}X\left(\frac{\omega}{a}\right), \quad a \neq 0} \tag{15}$$

Proof:
$$\begin{aligned}\int_{-\infty}^{\infty} x(\underbrace{at}_{\triangleq\tau})e^{-j\omega t}dt &= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau/a}\frac{d\tau}{a}, \text{ if } a > 0 \\ &= \int_{\infty}^{-\infty} x(\tau)e^{-j\omega\tau/a}\frac{d\tau}{a}, \text{ if } a < 0 \\ &= \frac{1}{|a|}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau/a}d\tau = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)\end{aligned}$$

Example 3 revisited: Applying (15) with $a = \frac{W}{\pi}$ to $x(t) = \text{sinc(t)}$,

$$x(at) = \text{sinc}\left(\frac{W}{\pi}t\right) \quad \overset{FT}{\leftrightarrow} \quad \frac{\pi}{W}X\left(\frac{\pi}{W}\omega\right)$$

where $X(\cdot)$ is as in (7). Thus,

$$\frac{W}{\pi}\text{sinc}\left(\frac{W}{\pi}t\right) \quad \overset{FT}{\leftrightarrow} \quad X\left(\frac{\pi}{W}\omega\right) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| \geq W, \end{cases}$$

which generalizes Example 3 to an arbitrary bandwidth W.

Special case of (15) with $a = -1$:

$$x(-t) \leftrightarrow X(-\omega) \tag{16}$$

$$\left.\begin{array}{ll} \text{If } x(-t) = x(t) \text{ then} & X(-\omega) = X(\omega) \\ \text{If } x(t) \text{ is also real:} & X(-\omega) = X^*(\omega) \end{array}\right\} \begin{array}{l} X(\omega) = X^*(\omega), \text{ i.e.,} \\ X(\omega) \text{is real.} \end{array}$$

Note that $X(\omega)$ is real in Examples 1 and 3 where x is real and even-symmetric.

Parseval's Relation:

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega} \tag{17}$$

Example: $\quad x(t) = e^{-at}u(t),\ a > 0 \ \leftrightarrow\ X(\omega) = \dfrac{1}{a + j\omega}$

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{0}^{\infty} e^{-2at} dt = \frac{1}{2a}\, e^{-2at}\Big|_{\infty}^{0} = \frac{1}{2a} \\ \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega &= \int_{-\infty}^{\infty} \frac{1}{a^2 + \omega^2} d\omega = \frac{1}{a}\ tan^{-1}\left(\frac{\omega}{a}\right)\Big|_{-\infty}^{\infty} = \frac{\pi}{a} = 2\pi \frac{1}{2a} \end{aligned}$$

Initial Value:

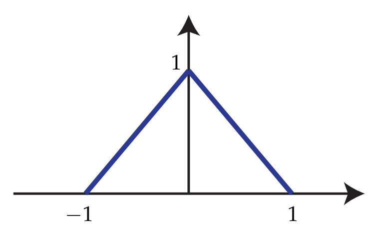
$$\boxed{x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) d\omega \qquad \text{(synthesis eq'n with } t = 0\text{)}} \tag{18}$$

DC Component:

$$\boxed{X(0) = \int_{-\infty}^{\infty} x(t) dt \qquad \text{(analysis equation with } \omega = 0\text{)}} \tag{19}$$

Convolution Property:

$$\boxed{(x_1 * x_2)(t) \quad \overset{FT}{\leftrightarrow} \quad X_1(\omega) X_2(\omega)} \tag{20}$$



Example: The triangular pulse shown on the right is the convolution of the rectangular pulse in Example 1 ($T_1 = 0.5$) with itself.

Thus, squaring the transform (8) and substituting $T_1 = 0.5$, we conclude that the Fourier Transform of the triangular pulse is:

$$\left(\text{sinc}\left(\frac{\omega}{2\pi}\right)\right)^2.$$