EE120 - Fall'19 - Lecture 4 Notes[1]

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Fourier Series for Continuous-Time Periodic Signals

Section 3.3 *in* Oppenheim & Willsky

Primer on Periodic Signals

A continuous-time signal x is periodic with period T if, for all t,

$$x(t+T) = x(t).$$

The smallest such T is called the *fundamental* period. Note that every integer multiple of the fundamental period is also a period.

Example: The signal

$$x(t) = \cos\left(\frac{2\pi}{3}t\right) + \sin\left(\frac{\pi}{10}t\right)$$

is periodic with fundamental period $T = 60$, which is the smallest value that is an integer multiple of both 3 and 20, the fundamental periods of the two terms.

Question: Is the sum of two periodic signals periodic?

The answer is *not necessarily*. Given two periods $T_1, T_2 \neq 0$, a common period T must satisfy $T = n_1 T_1 = n_2 T_2$ for some integers n_1, n_2. Such integers exists if and only if T_1/T_2 is rational. The sum in the example above is periodic, while the signal $\cos(2\pi t) + \cos(t)$ is not.

Using complex exponentials to represent sinusoidal signals greatly simplifies the algebra involved in Fourier Series discussed below.

Example: We can rewrite $a\cos(\omega t) + b\sin(\omega t)$ as

$$\frac{a}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) + \frac{b}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) = \frac{a-bj}{2}e^{j\omega t} + \frac{a+bj}{2}e^{-j\omega t}. \quad (1)$$

Likewise we can represent $\cos(\omega t + \phi)$ with phase ϕ as

$$\cos(\omega t + \phi) = \frac{e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}}{2} = \frac{e^{j\phi}}{2}e^{j\omega t} + \frac{e^{-j\phi}}{2}e^{-j\omega t}. \quad (2)$$

Note that in (1) and (2) the coefficients of $e^{j\omega t}$ and $e^{-j\omega t}$ are complex conjugates. This is because the signal in each case is real-valued. Indeed, for a signal of the form $x(t) = ce^{j\omega t} + de^{-j\omega t}$ to be real we need $x(t) = x^*(t) = c^* e^{-j\omega t} + d^* e^{j\omega t}$ for all t, which requires $c = d^*$.

Thus the *negative frequency* term $e^{-j\omega t}$ in (1) and (2) is an artifact of the complex exponential representation of real-valued signals. Its role is to cancel out the imaginary terms due to $e^{j\omega t}$ and its coefficient.

Fourier Series

Joseph Fourier (1768-1830)

Fourier Series represents a periodic signal with fundamental period T as a weighted sum of sinusoidals $e^{jk\omega_0 t}$ $k = 0, \mp 1, \mp 2, ...,$ where

$$\omega_0 = \frac{2\pi}{T}$$

is called the fundamental frequency. The series thus has the form

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.} \tag{3}$$

This is called the "synthesis equation," as it synthesizes a periodic signal from sinusoidal components. The term $k = 0$ (often called the "DC component" in reference to direct current circuits) is constant:

$$a_k e^{jk\omega_0 t}|_{k=0} = a_0.$$

The terms $k = \mp 1$ are known as the "first harmonic," since they oscillate at the fundamental frequency ω_0. Likewise $k = \mp 2$ terms oscillate at $2\omega_0$ and are referred to as the "second harmonic."

Example: $$x(t) = 1 + \underbrace{\frac{1}{2}\cos(2\pi t)}_{= \frac{1}{4}e^{j2\pi t} + \frac{1}{4}e^{-j2\pi t}} + \underbrace{\sin(4\pi t)}_{= \frac{1}{2j}e^{j4\pi t} - \frac{1}{2j}e^{-j4\pi t}} + \underbrace{\frac{2}{3}\cos(6\pi t)}_{= \frac{1}{3}e^{j6\pi t} + \frac{1}{3}e^{-j6\pi t}}$$

can be written in the form (3) with $a_0 = 1$, $a_1 = a_{-1} = \frac{1}{4}$, $a_2 = -a_{-2} = \frac{1}{2j}$, $a_3 = a_{-3} = \frac{1}{3}$. In this example we don't need any terms beyond the third harmonic $k = \mp 3$.

Conjugate Symmetry Property: If $x(t)$ has Fourier series coefficients a_k, then $x^*(t)$ has Fourier series coefficients $b_k = a^*_{-k}$.

This follows by taking the complex conjugate of both sides of (3). If x is real-valued then $x(t) = x^*(t)$ for all t and, thus, $a_k = b_k = a^*_{-k}$.

Corollary: If x is real-valued, then

$$a_k = a^*_{-k}. \tag{4}$$

How to Find the Fourier Series Coefficients a_k in General?

Multiply both sides of the synthesis equation (3) with $e^{-jn\omega_0 t}$ and integrate from 0 to $T = \frac{2\pi}{\omega_0}$:

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \underbrace{\left(\int_0^T e^{j(k-n)\omega_0 t}dt\right)}_{= \begin{cases} T & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}} = Ta_n.$$

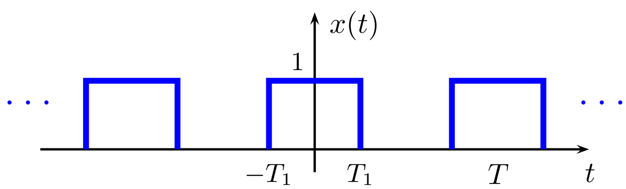
We thus have the following "analysis equation" to find the Fourier Series coefficients for any periodic signal x:

$$\boxed{a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt.} \tag{5}$$

In particular, the DC component $a_0 = \frac{1}{T}\int_0^T x(t)dt$ is the average of $x(t)$ over one period.

Note that, since the integrand in (5) is periodic, we can perform the integral over any period $[t_0, t_0+T]$ and obtain the same result as $[0, T]$. In the examples below we integrate over $[-T/2, T/2]$ to take advantage of the even symmetry of the signals.

Example: Periodic Square Wave



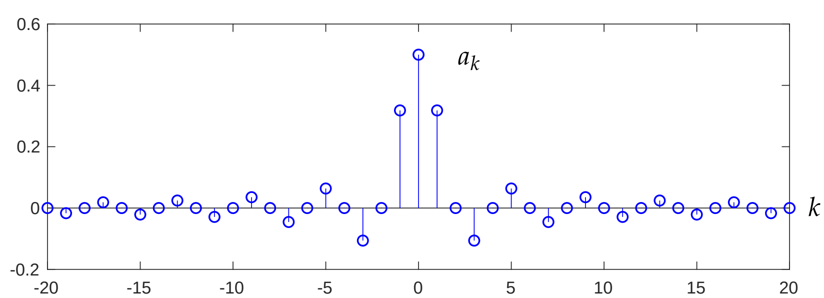
For $k = 0$,

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}.$$

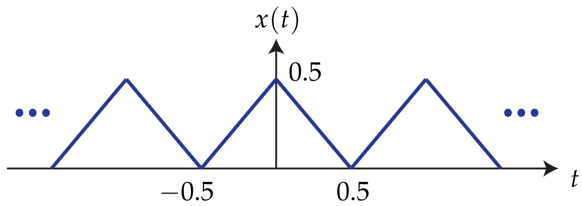
For $k \neq 0$,

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt = \frac{1}{T}\frac{-1}{jk\omega_0}\underbrace{e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}}_{\substack{= e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1} \\ = -2j\sin(k\omega_0 T_1)}} \\ &= \frac{2}{k\omega_0 T}\sin(k\omega_0 T_1) = \frac{1}{k\pi}\sin\left(2\pi k\frac{T_1}{T}\right). \end{aligned} \tag{6}$$

Below is a plot of the coefficients a_k from $k = -20$ to $k = 20$ when $T_1 = T/4$.



Example: Periodic Triangle Wave



This signal is periodic with $T = 1$ and is expressed by

$$x(t) = 0.5 - |t| \quad \text{when } t \in [-0.5, 0.5].$$

It is easy to see that $a_0 = 0.25$, since the average over one period is the area of each triangle. For $k \neq 0$, we need to perform the integral

$$\begin{aligned} a_k &= \int_{-0.5}^{0.5} (0.5 - |t|) e^{-j2\pi kt} dt \\ &= \underbrace{\int_{-0.5}^{0.5} 0.5 e^{-j2\pi kt} dt}_{= 0 \text{ (show this)}} + \underbrace{\int_{-0.5}^{0} t e^{-j2\pi kt} dt}_{\text{define as } \alpha(k)} - \underbrace{\int_{0}^{0.5} t e^{-j2\pi kt} dt.}_{= -\alpha(-k) \text{ (show)}} \end{aligned}$$

It follows from integration by parts that

$$\alpha(k) = \frac{1}{4\pi^2 k^2} \left(1 + j\pi k\, e^{j\pi k} - e^{j\pi k} \right)$$

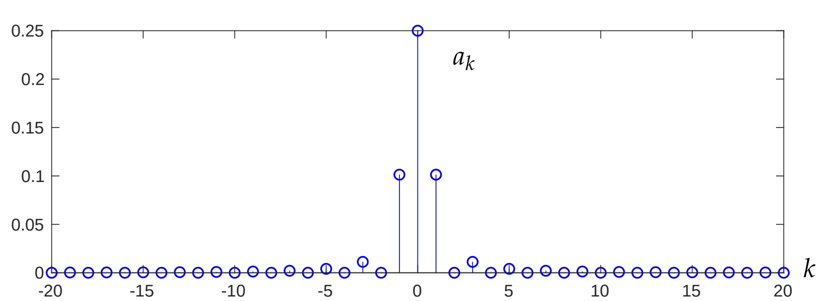
and $a_k = \alpha(k) + \alpha(-k)$ simplifies to

$$\begin{aligned} a_k &= \frac{1}{2\pi^2 k^2} + \frac{1}{4\pi^2 k^2} j\pi k \underbrace{(e^{j\pi k} - e^{-j\pi k})}_{\substack{= 2j\sin(\pi k) \\ = 0 \text{ for integer } k}} - \frac{1}{4\pi^2 k^2} \underbrace{(e^{j\pi k} + e^{-j\pi k})}_{= 2\cos(\pi k)} \\ &= \frac{1}{2\pi^2 k^2} (1 - \cos(\pi k)). \end{aligned}$$

Since $1 - \cos(\pi k) = 0$ when k is even and $= 2$ when k is odd, we further simplify this expression and combine with $a_0 = 0.25$:

$$a_k = \begin{cases} 0.25 & k = 0 \\ 0 & k \text{ even and } \neq 0 \\ \frac{1}{\pi^2 k^2} & k \text{ odd.} \end{cases} \tag{7}$$

Below is a plot of the coefficients a_k from $k = -20$ to $k = 20$.



A Convergence Result

A natural question is whether the Fourier Series converges to signal x, as implicitly assumed in (3). That is, if we define the partial sum

$$x_M(t) = \sum_{k=-M}^{M} a_k e^{jk\omega_0 t} \tag{8}$$

and take a point τ in time, does $\lim_{M\to\infty} x_M(\tau)$ exist and equal $x(\tau)$?

The following result[2] establishes such convergence when x and its derivative are piecewise continuous[3]:

[2] by Gustav Dirichlet (1805-1859)

[3] continuous except at a finite number of points

Theorem: Suppose x is piecewise continuous with piecewise continuous derivative, and periodic with fundamental period T and frequency $\omega_0 = 2\pi/T$. If x is continuous at $t = \tau$, then

$$\lim_{M\to\infty} x_M(\tau) = x(\tau).$$

If x is discontinuous at $t = \tau$ with left and right limits $x(\tau^-)$, $x(\tau^+)$,

$$\lim_{M\to\infty} x_M(\tau) = \frac{1}{2}\left(x(\tau^-) + x(\tau^+)\right).$$

For the continuous triangle wave example above, the theorem states that the Fourier Series converges at each point (Figure 1).

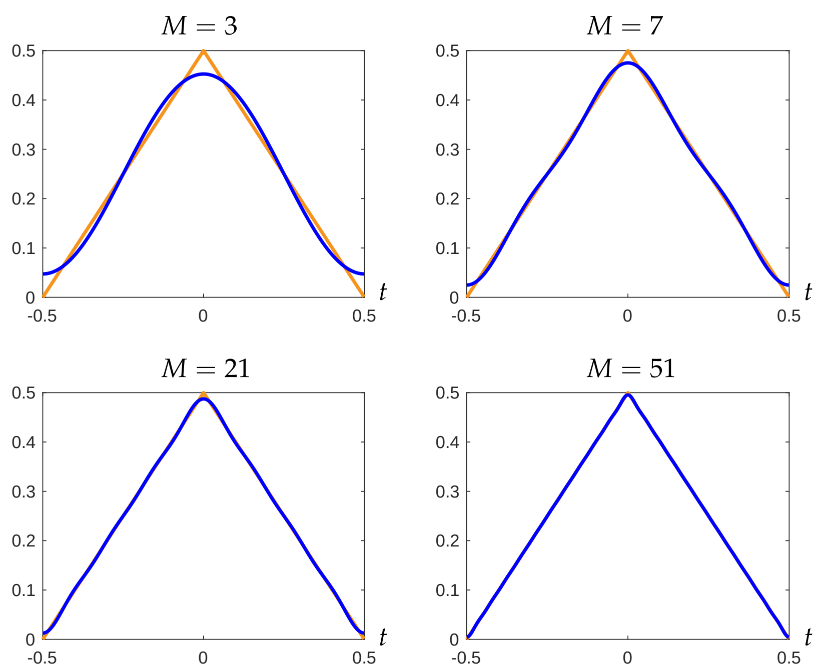


Figure 1: The partial sum (8) with the Fourier Series coefficients (7) for the triangle wave. As M is increased the sum converges at each point to the triangle wave.

For the square wave example, the Fourier Series converges to 1 or 0 away from the jumps, and to their average 0.5 at the jumps (Figure 2).

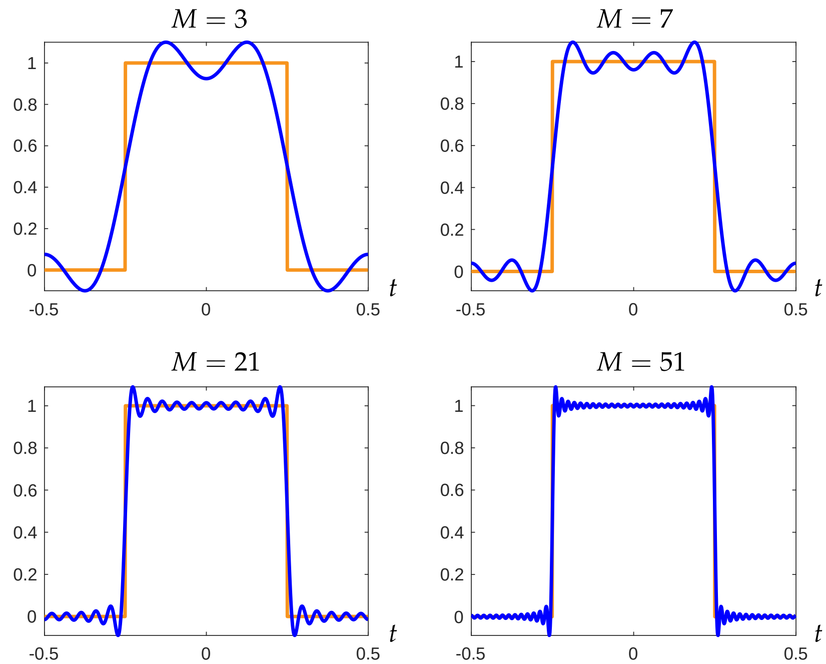


Figure 2: The partial sum (8) with the Fourier Series coefficients (6) for the square wave with $T = 1$, $T_1 = 0.25$. The sum converges to 1 or 0 away from the jumps, and to 0.5 at the jumps.

The oscillatory behavior near the jumps in Figure 2 is known as the Gibbs[4] Phenomenon. The ripples become more compressed with increasing M, but they don't disappear. In fact an overshoot of about 18% remains no matter how large we select M.

[4] named after J.W. Gibbs (1839-1903)

Gibbs Phenomenon does not contradict the theorem above, which claims convergence for any point in time, not convergence of the graph of x_M to that of x. Pointwise convergence indeed occurs in Figure 2: if we fix a point $t = \tau$ near a jump, the value $x_M(\tau)$ will converge to 1 or 0 as $M \to \infty$, since the ripples compress and move closer to the jump.

Properties of Fourier Series (FS)

The following are easy to derive from the analysis equation (5):

1. Linearity: If two signals x, y with identical periods have FS coefficients a_k, b_k, then $Ax + By$ has FS coefficients $Aa_k + Bb_k$.

2. Time shift: If x has FS coefficients a_k, then $\hat{x}(t) = x(t - t_0)$ has FS coefficients $a_k e^{-jk\omega_0 t_0}$.

3. Time reversal: If x has FS coefficients a_k, then $\hat{x}(t) = x(-t)$ has FS coefficients a_{-k}.

Thus, if x is even symmetric ($x(t) = x(-t)$), then $a_k = a_{-k}$. When x is real we combine this with (4) and conclude $a_k = a_{-k} = a_k^*$, that is, each a_k is real.

Corollary: FS coefficients of a real, even symmetric signal are real.