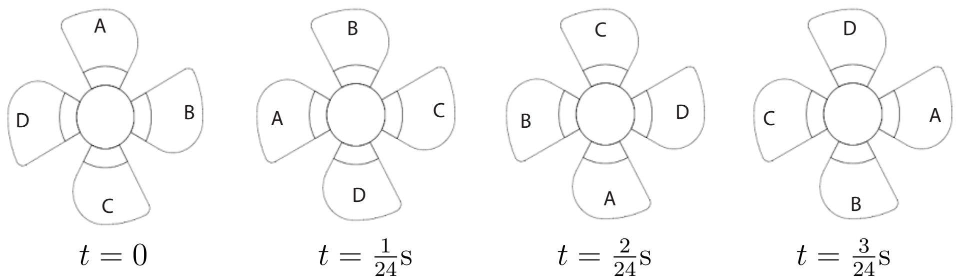
EE 120 SIGNALS AND SYSTEMS, Fall 2019
Homework # 4, Due November 1, Friday

1. A fan with four blades is rotating clockwise. When recorded with a video camera at 24 frames per second, the blades appear to be moving counterclockwise as shown below. List all possible rates at which the fan may be rotating in revolutions per second.

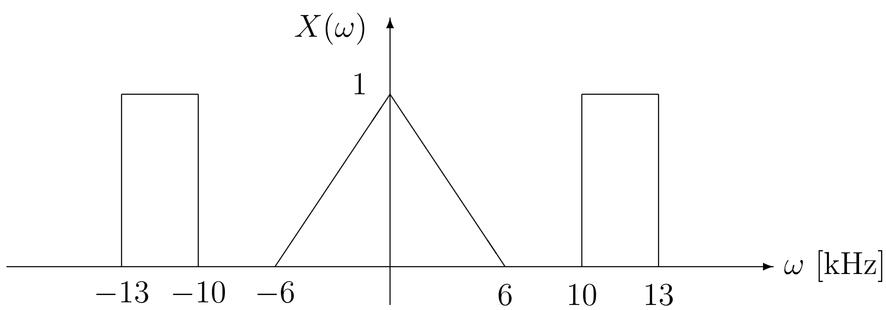


2. A continuous-time signal x, with spectrum $X(\omega)$ below, is sampled with period T:

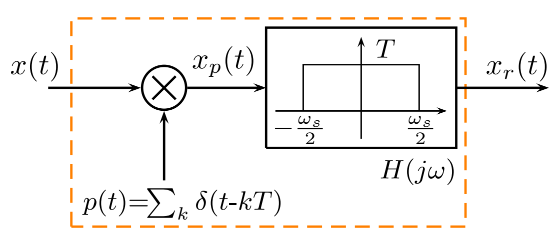
$$x_p(t) = x(t)p(t), \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{1}$$

a) What is the minimum sampling rate to recover the original signal x from x_p? (Hint: the gap from 6 to 10 kHz in $X(\omega)$ allows a lower frequency than the Nyquist rate.)

b) For the sampling rate determined in part (a) plot $X_p(\omega)$ and the frequency response of the reconstruction filter that must be used to recover x.



3. In this problem we show that sampling followed by sinc interpolation (*i.e.*, low pass filtering) is linear but not time-invariant.



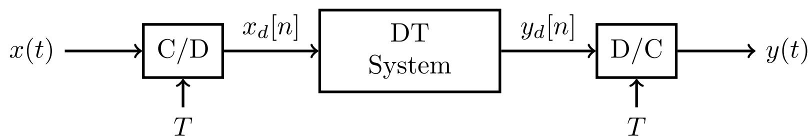
a) Show that the sampling operation (1) is linear with x as input and x_p as output. Since the low-pass filter is also linear, the combined system $x \mapsto x_r$ is linear.

b) Let $T = 1$ and $x(t) = \cos(2\pi t)$. What is $x_r(t)$?

c) Let $T = 1$ and input $\hat{x}(t) = x(t - \frac{1}{2}) = \cos\left(2\pi(t - \frac{1}{2})\right)$. What is the output $\hat{x}_r(t)$? Is $\hat{x}_r(t) = x_r(t - \frac{1}{2})$?

Note that if we restrict the input space to signals with bandwidth $\omega_M < \pi/T$, then $x_r = x$ and we have both linearity and time-invariance.

4. Consider the scheme shown below for discrete-time processing of continuous-time signals and suppose we wish to emulate a continuous time differentiator; that is, the desired frequency response if $H(\omega) = j\omega$.



If we design a discrete-time LTI system with frequency response:

$$H_d(e^{j\Omega}) = j\Omega/T \quad |\Omega| < \pi \tag{2}$$

then the effective frequency response from a continuous-time input x with bandwidth $\omega_M < \pi/T$ to the continuous-time output y matches the desired response $H(\omega)$:

$$H_d(e^{j\Omega})\big|_{\Omega=\omega T} = j\omega.$$

a) Find the impulse response of the discrete-time system with frequency response (2).

b) Suppose we truncate the impulse response in part (a) for $|n| \geq 2$ and implement the FIR filter with impulse response:

$$\hat{h}_d[n] = \begin{cases} \alpha h_d[n] & n = -1, 0, 1 \\ 0 & |n| \geq 2. \end{cases}$$

Calculate the frequency response $\hat{H}_d(e^{j\Omega})$ and select the constant α such that the first term in the Taylor approximation of $\hat{H}_d(e^{j\Omega})$ about $\Omega = 0$ matches $H_d(e^{j\Omega})$. Provide a plot superimposing $|\hat{H}_d(e^{j\Omega})|$ and $|H_d(e^{j\Omega})|$ for $|\Omega| < \pi$.

c) Write a difference equation relating the input $x_d[n]$ and output $y_d[n]$ of the truncated filter $\hat{H}_d(e^{j\Omega})$ in part (b).

5. Given the 2D signal $x[n_1, n_2]$ and sampling periods N_1, N_2, define the impulse train sampling:

$$x_p[n_1, n_2] \triangleq x[n_1, n_2] \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \delta[n_1 - k_1 N_1, n_2 - k_2 N_2].$$

a) Write a formula for $X_p(e^{j\omega_1}, e^{j\omega_2})$ in terms of $X(e^{j\omega_1}, e^{j\omega_2})$. Your answer should mimic the 1D formula in Lecture 14 (no need to derive).

b) How should the bandwidth of $x[n_1, n_2]$ be restricted to avoid aliasing?

c) What is the impulse response of the ideal reconstruction filter that recovers $x[n_1, n_2]$ from $x_p[n_1, n_2]$ when there is no aliasing?