EE 120 SIGNALS AND SYSTEMS, Fall 2019
Homework # 3, Due October 24, Thursday

1. Before taking the derivatives of a signal $x(t)$ it is customary to low-pass filter it so that any noise present in the signal is not further amplified by differentiation.

   a) Let the low-pass filter have impulse response $h(t)$ and show that the kth derivative of the filtered signal $(h * x)(t)$ can be written as:

$$\frac{d^k}{dt^k}(h * x)(t) = (h_k * x)(t)$$

   where you should express $h_k(t)$ in terms of $h(t)$. Thus, low-pass filtering followed by k derivatives can be combined into a single convolution with an appropriate h_k.

   b) Suppose we select a low-pass filter with Gaussian impulse response:

$$h(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}.$$

   Find $h_1(t)$, its Fourier Transform $H_1(\omega)$, and make a rough plot of $|H_1(\omega)|$.

   c) Find the frequency response $H_{\text{diff}}(\omega)$ of the differentiator $y(t) = \frac{d}{dt}x(t)$ without low-pass filtering. How does $|H_{\text{diff}}(\omega)|$ compare with $|H_1(\omega)|$ in part (b)?

2. (All-Pass System) Consider an LTI system with frequency response:

$$H(e^{j\omega}) = \frac{e^{-j\omega} - a}{1 - ae^{-j\omega}}$$

   where a is a real number such that $|a| < 1$.

   a) Show that $|H(e^{j\omega})| = 1$ at all frequencies.

   b) Derive an expression for the phase $\angle H(e^{j\omega})$.

   c) Take $a = 1/\sqrt{3}$ and determine the output $y[n]$ for the input:

$$x[n] = \cos\left(\frac{\pi}{6}n\right) + \cos(\pi n).$$

   d) Write a difference equation that implements a LTI system with the frequency response above.

3. Suppose we apply the length-100 sequence

$$x[n] = \cos(0.1\pi n) + 0.5\cos(\pi n) \quad n = 0, 1, \ldots, 99$$

   to a LTI system whose impulse response is

$$h[n] = \begin{cases} \frac{1}{4} & n = 0, 1, 2, 3 \\ 0 & \text{otherwise.} \end{cases}$$

   a) What is the length N of the output signal $y = h * x$?

   b) Use the convolution property of DFT (Lecture 10) to compute and plot the output sequence $y[n]$, $n = 0, \ldots, N-1$. You can use Python or any other software that has functions for FFT and inverse FFT. Attach your code and plot.

4. a) Find the impulse response of the 2D filter whose output is $y[n_1, n_2] =$

$$\frac{1}{5}\left(x[n_1, n_2] + x[n_1-1, n_2] + x[n_1+1, n_2] + x[n_1, n_2-1] + x[n_1, n_2+1]\right).$$

   b) Calculate the frequency response $H(e^{j\omega_1}, e^{j\omega_2})$ and plot in the range $(\omega_1, \omega_2) \in [-\pi, \pi] \times [-\pi, \pi]$ using appropriate software (e.g., MATLAB command `surf` or Python command `plot_surface`). Attach your code and plot.

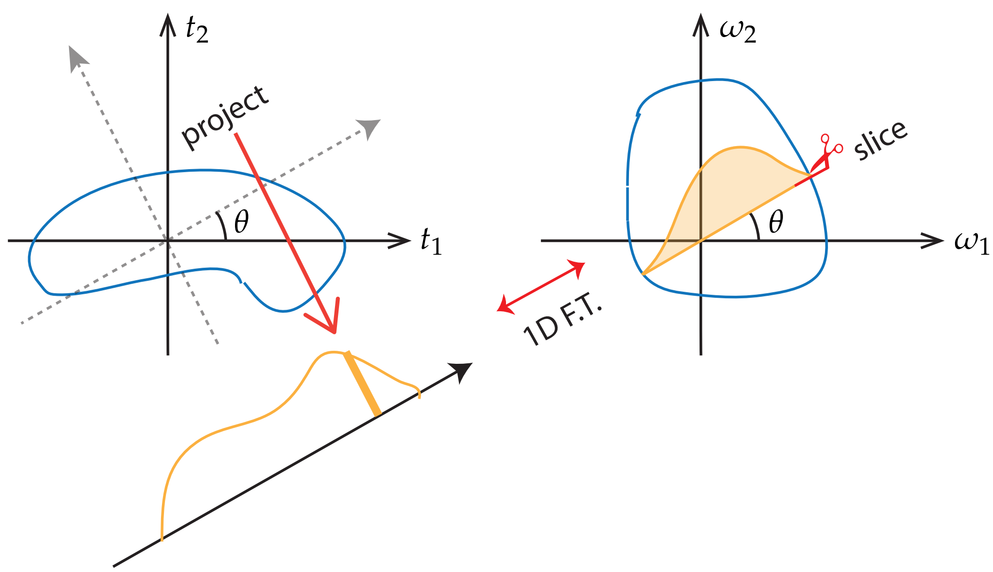
5. (Projection-Slice Theorem) Consider the following "projection" of the 2D function $x(t_1, t_2)$ along the t_2 axis:

$$x_0(t_1) \triangleq \int_{-\infty}^{\infty} x(t_1, t_2) dt_2.$$

   Show that the 1D CTFT of $x_0(t_1)$ is related to the 2D CTFT of $x(t_1, t_2)$ by:

$$X_0(j\omega_1) = X(j\omega_1, j\omega_2)\big|_{\omega_2=0}.$$

   A generalization of this property to projections along any direction is known as the Projection-Slice Theorem and is illustrated in the figure below. You are not asked to derive this generalization, but note that projection along the t_2 axis above is the special case $\theta = 0$ in the figure.



   The Projection-Slice Theorem is crucial in tomography where one collects projections at many angles about a 2D object. The 1D Fourier Transform of each such "shadow" corresponds to a slice of the 2D Fourier Transform. One can thus obtain the 2D Fourier Transform by combining these slices and then reconstruct $x(t_1, t_2)$ from the synthesis equation.