EE 120 SIGNALS AND SYSTEMS, Fall 2019
Homework # 2, Due September 26, Thursday

1. In class we showed that the Fourier Series coefficients of the square wave

$$x(t) = \begin{cases} 1 & |t| \leq T_1 \\ 0 & T_1 < |t| \leq T/2, \end{cases}$$

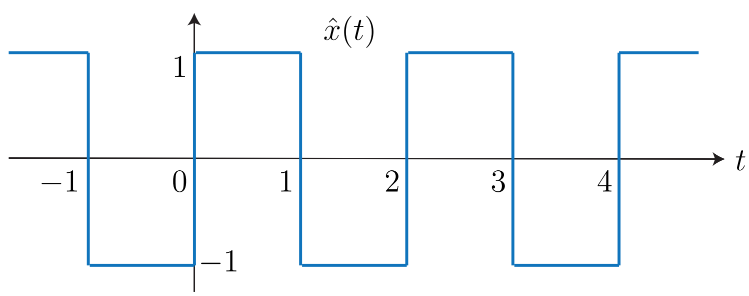
repeated with period T, are:

$$a_k = \begin{cases} \frac{2T_1}{T} & k = 0 \\ \frac{1}{k\pi} \sin\left(2\pi k \frac{T_1}{T}\right) & k \neq 0. \end{cases}$$

Using this expression and the linearity and time shift properties of the Fourier Series (Lecture 4), find the Fourier Series coefficients of the signal $\hat{x}$ below. You should be able to simplify your answer to the form:

$$\hat{a}_k = \begin{cases} \_\_\_\_\_ & k : \text{even} \\ \_\_\_\_\_ & k : \text{odd}, \end{cases}$$

where no sine function is needed for either odd or even k.



2. Prove the following properties of the Fourier Series coefficients where, in each case, the signal x is assumed to be periodic with period T:

a) If $x(t) = x(-t)$ for all t, then each a_k is purely real.

b) If $x(t) = -x(-t)$ for all t, then each a_k is purely imaginary.

c) If $x(t) = -x(t + T/2)$ for all t, then $a_k = 0$ for every even k.

d) Determine which of the symmetry properties in parts a,b,c are satisfied by the signals x and $\hat{x}$ in Problem 1 and confirm the corresponding properties of the coefficients a_k and $\hat{a}_k$.

3. a) Consider a period-N signal $x[n]$ and recall the Fourier series expansion:

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \quad \omega_0 = 2\pi/N.$$

Rewrite this expression as

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \mathbf{W} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix}$$

where $\mathbf{W}$ is a $N \times N$ matrix you should determine.

b) Apply the matrix equation you derived in part (a) to the case where $x[n]$ is the impulse train with period $N = 3$. Using the resulting three equations determine the three unknowns a_0,a_1,a_2 by using any standard technique for solving linear systems of equations.

c) Generalizing the example in part (b), note that the Fourier series coefficients $a_0, \cdots, a_{N-1}$ can be calculated from:

$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix} = \mathbf{W}^{-1} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

where $\mathbf{W}$ is the matrix you found in part (a). Find $\mathbf{W}^{-1}$ without actually calculating the inverse, but using the analysis equation derived in class.

d) Write $\mathbf{W}$ from part (a) and $\mathbf{W}^{-1}$ from part (c) explicitly for $N = 3$, and multiply them to verify that $\mathbf{W}\mathbf{W}^{-1} = I$.

4. In this problem you will show that the Fourier Transform of a Gaussian function $x(t) = e^{-t^2}$ is itself (an appropriately scaled) Gaussian function. To get started note that

$$X(\omega) = \int_{-\infty}^{\infty} e^{-t^2} e^{-j\omega t} dt$$

satisfies

$$X(0) = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi} \qquad (1)$$

from the well-known Gaussian integral formula, and

$$\frac{dX(\omega)}{d\omega} = \int_{-\infty}^{\infty} -jte^{-t^2} e^{-j\omega t} dt = \frac{j}{2} \int_{-\infty}^{\infty} \frac{d}{dt} \left(e^{-t^2} \right) e^{-j\omega t} dt. \qquad (2)$$

a) Use integration by parts in (2) to show that

$$\frac{dX(\omega)}{d\omega} = -\frac{\omega}{2} X(\omega).$$

b) Show that $X(\omega) = \alpha e^{-\omega^2/\beta}$ satisfies this differential equation subject to the initial condition (1) with a choice of α and β you should determine.

5. Let $s(t)$ be a real-valued signal for which $S(j\omega) = 0$ when $|\omega| > \omega_0$. Amplitude modulation is performed to produce the signal:

$$r(t) = s(t)\cos(\omega_0 t + \phi)$$

and the demodulation scheme depicted below is applied to $r(t)$ at the receiver end. Determine $y(t)$ assuming that the ideal lowpass filter has a cutoff frequency of ω_0 and a passband gain of 2. What happens when the phase difference between the receiver and transmitter is $\phi = 90°$?