

---

---

# CHAPTER 1

---

## 2021-2022 学年微积分（一）（下）期中考试

### 1 基本计算题 (每小题 6 分, 共 60 分)

1. 求微分方程  $y'' + 9y = x \cos 3x$  对应齐次方程的通解, 并写出非齐次特解的待定特解形式.
2. 设二阶线性微分方程  $y'' + a(x)y' + b(x)y = f(x)$  有三个特解  $y_1 = x$ ,  $y_2 = x + 2e^x$ ,  $y_3 = x + (2 + 3x)e^x$ , 求其通解.
3. 已知两直线  $L_1 : \begin{cases} x - 3y + z = 0, \\ 2x - 4y + z = -1 \end{cases}$  和  $L_2 : x = \frac{y+1}{3} = \frac{z-2}{4}$ , 求  $L_1$  与  $L_2$  之间的距离  $d$ .

4. 设由方程  $F(x-y, y-z, z-x) = 0$  可以确定隐函数  $z = z(x, y)$ ,  $F$  具有连续的偏导数, 且  $F'_2 - F'_3 \neq 0$ , 求  $dz$ .

5. 求曲线  $L: \begin{cases} x^2 + y^2 + z^2 = 4, \\ (x-1)^2 + y^2 = 1 \end{cases}$  在点  $P(1, 1, \sqrt{2})$  处的法平面方程.

6. 求椭圆曲线  $\begin{cases} z = x^2 + y^2, \\ x + y + z = 4 \end{cases}$  上距离原点最近的点.

7. 计算  $I = \iint_D (x^2 y^2 + x \sin(x^2 + y^2)) \, dx \, dy$ , 其中  $D = \{(x, y) | |x| + |y| \leq 1\}$ .

8. 设平面区域  $D$  由直线  $y = x$ , 圆弧  $y = 1 + \sqrt{1 - x^2}$  及  $y$  轴所围成, 计算  $I = \iint_D xy \, d\sigma$ .

9. 求二次积分  $I = \int_0^1 dx \int_x^1 e^{-y^2} dy$ .

10. 求  $I = \iiint_{\Omega} \frac{1}{\sqrt{x^2 + y^2 + z^2}} dv$ , 其中  $\Omega$  是由曲面  $z = \sqrt{x^2 + y^2}$  与  $z = 1$  所围成的区域.

## 2 综合题 (每小题 8 分, 共 40 分)

11. 设  $z$  具有二阶连续偏导数, 已知变换  $\begin{cases} u = x - 2\sqrt{y}, \\ v = x + 2\sqrt{y} \end{cases}$  将方程  $\frac{\partial^2 z}{\partial x^2} - y \frac{\partial^2 z}{\partial y^2} - \frac{1}{2} \frac{\partial z}{\partial y} = 0$  化为以  $u, v$  为自变量的方程, 求新方程形式.

12. 讨论二元函数  $f(x, y) = \begin{cases} \frac{x^2 y^2}{(x^2 + y^2)^{\frac{3}{2}}}, & x^2 + y^2 \neq 0, \\ 0, & x^2 + y^2 = 0 \end{cases}$  在原点  $(0, 0)$  处的连续性、偏导数存在性及可微性.

13. 求常数  $a, b, c$  的值, 使得函数  $f(x, y, z) = axy^2 + byz + cx^3z^2$  在点  $M(1, 2, -1)$  处的所有方向导数中, 沿  $x$  轴正向的方向导数最大, 且该最大值为 64.

14. 求  $I = \iiint_{\Omega} z \, dv$ , 其中  $\Omega$  是由平面曲线  $\begin{cases} y^2 = 2z, \\ x = 0 \end{cases}$  绕  $z$  轴旋转一周所得的旋转曲面与平面  $z = 1$ ,  $z = 2$  所围成的区域.

15. 已知曲面  $x^2 + y^2 + z = 4$  将球体  $x^2 + y^2 + z^2 \leq 4z$  分为两部分, 求这两部分的体积比.

---

## CHAPTER 2

---

### 2021-2022 学年微积分（一）（下）期中考试参考答案

#### 1 基本计算题 (每小题 6 分, 共 60 分)

1. **Solution.** 方程对应的齐次方程为  $y'' + 9y = 0$ , 其特征方程为  $r^2 + 9 = 0$ , 解得  $r = \pm 3i$ ,

所以齐次方程的通解为  $C_1 \cos 3x + C_2 \sin 3x$ .

对于  $y = x \cos 3x$ ,  $\lambda = 3i$  是特征方程的单根,

所以非齐次特解的待定特解形式为  $y^* = x[(Ax + B) \cos 3x + (Cx + D) \sin 3x]$ ,

其中  $A, B, C, D$  为待定常数.

2. **Solution.** 由题意可知  $y_2 - y_1 = 2e^x$ ,  $y_3 - y_2 = 3xe^x$  是齐次方程  $y'' + a(x)y' + b(x)y = 0$  的两个特解, 将其代入, 得

$$\begin{cases} 2e^x + 2a(x)e^x + 2b(x)e^x = 0, \\ 3(x+2)e^x + 3a(x)(x+1)e^x + 3xb(x)e^x = 0. \end{cases}$$

解得  $\begin{cases} a(x) \equiv -2, \\ b(x) \equiv 1. \end{cases}$  所以齐次方程为  $y'' - 2y' + y = 0$ ,

对应的特征方程为  $r^2 - 2r + 1 = 0$ , 解得二重根  $r = 1$ , 因此通解为  $(Ax + B)e^x$ ,

故原方程的通解为  $y = (Ax + B)e^x + x$ , 其中  $A, B$  为任意常数.

3. **Solution.** 取直线  $L_1$  的方向向量  $\mathbf{s}_1 = (1, -3, 1) \times (2, -4, 1) = (1, 1, 2)$ , 直线  $L_2$  的方向向量  $\mathbf{s}_2 = (1, 3, 4)$ .

在方程组  $\begin{cases} x - 3y + z = 0, \\ 2x - 4y + z = -1 \end{cases}$  中令  $y = 0$ , 取  $L_1$  上的点  $P(-1, 0, 1)$ , 取  $L_2$  上的点  $Q(0, -1, 2)$ .

平行六面体  $\overrightarrow{PQ}\mathbf{s}_1\mathbf{s}_2$  的体积  $V = \left| \overrightarrow{PQ} \cdot (\mathbf{s}_1 \times \mathbf{s}_2) \right| = \left| \begin{vmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & 4 \end{vmatrix} \right| = 2$ ,

又  $V = h \cdot |\mathbf{s}_1 \times \mathbf{s}_2|$ , 其中  $h$  为两直线之间的距离, 所以  $h = \frac{V}{|\mathbf{s}_1 \times \mathbf{s}_2|} = \frac{2}{2\sqrt{3}} = \frac{\sqrt{3}}{3}$ .

4. **Solution.** 方程  $F(x-y, y-z, z-x) = 0$  两边全微分, 得

$$F'_1(dx - dy) + F'_2(dy - dz) + F'_3(dz - dx) = 0,$$

$$\text{解得 } dz = \frac{F'_1 - F'_3}{F'_2 - F'_3} dx + \frac{F'_2 - F'_1}{F'_2 - F'_3} dy.$$

5. **Solution.** 设  $F(x, y, z) = x^2 + y^2 + z^2 - 4$ ,  $G(x, y, z) = (x-1)^2 + y^2 - 1$ .

$$\text{法一. 因 } \left. \frac{\partial(F, G)}{\partial(y, z)} \right|_P = \begin{vmatrix} 2y & 2z \\ 2y & 0 \end{vmatrix}_P = -4yz|_{(1, 1, \sqrt{2})} = -4\sqrt{2} \neq 0,$$

所以方程组  $\begin{cases} F(x, y, z) = 0, \\ G(x, y, z) = 0 \end{cases}$  可以唯一确定两个具有连续导数的隐函数  $y = y(x), z = z(x)$ .

$$\text{方程组 } \begin{cases} F(x, y, z) = x^2 + y^2 + z^2 - 4 = 0, \\ G(x, y, z) = (x-1)^2 + y^2 - 1 = 0 \end{cases} \quad \text{两边对 } x \text{ 求导, 得}$$

$$\begin{cases} 2x + 2y \frac{dy}{dx} + 2z \frac{dz}{dx} = 0, \\ 2(x-1) + 2y \frac{dy}{dx} = 0. \end{cases}$$

$$\text{解得 } \begin{cases} \frac{dy}{dx} = \frac{1-x}{y}, \\ \frac{dz}{dx} = -\frac{1}{z} \end{cases}, \text{ 所以法平面的法向量可取为 } \sqrt{2} \left( 1, \frac{dy}{dx}, \frac{dz}{dx} \right)_P = (\sqrt{2}, 0, -1),$$

法平面的方程为  $\sqrt{2} \cdot (x-1) - 0 \cdot (y-1) + (-1) \cdot (z - \sqrt{2}) = 0$ , 即  $\sqrt{2}x - z = 0$ .

法二.  $\nabla F = (2x, 2y, 2z)$ ,  $\nabla G = (2(x-1), 2y, 0)$ ,

取两个曲面的法向量分别为  $\mathbf{n}_F = \frac{1}{2} \nabla F|_P = (1, 1, \sqrt{2})$ ,  $\mathbf{n}_G = \frac{1}{2} \nabla G|_P = (0, 1, 0)$ ,

$$\text{所以法平面的法向量可取为 } \mathbf{n}_F \times \mathbf{n}_G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & \sqrt{2} \\ 0 & 1 & 0 \end{vmatrix} = (-\sqrt{2}, 0, 1),$$

法平面的方程为  $-\sqrt{2} \cdot (x-1) - 0 \cdot (y-1) + 1 \cdot (z - \sqrt{2}) = 0$ , 即  $\sqrt{2}x - z = 0$ .

6. **Solution.** 设  $P(x, y, z)$  为椭圆曲线上的任意一点,

即在约束条件  $\begin{cases} z = x^2 + y^2, \\ x + y + z = 4 \end{cases}$  下, 求距离平方函数  $x^2 + y^2 + z^2$  的最小值.

用 Lagrange 乘数法, 设  $L(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 + \lambda(x^2 + y^2 - z) + \mu(x + y + z - 4)$ , 则

$$\begin{cases} \frac{\partial L}{\partial x} = 2x + 2\lambda x + \mu = 0, \\ \frac{\partial L}{\partial y} = 2y + 2\lambda y + \mu = 0, \\ \frac{\partial L}{\partial z} = 2z - \lambda + \mu = 0, \\ \frac{\partial L}{\partial \lambda} = x^2 + y^2 - z = 0, \\ \frac{\partial L}{\partial \mu} = x + y + z - 4 = 0. \end{cases}$$

$$\frac{\partial L}{\partial x} - \frac{\partial L}{\partial y} = 2(\lambda + 1)(x - y) = 0, \text{ 若 } \lambda = -1, \text{ 则 } \frac{\partial L}{\partial x} = \mu = 0,$$

$$\frac{\partial L}{\partial z} = 2z - \lambda + \mu = 2z + 1 = 0, \text{ 解得 } z = -\frac{1}{2}, \text{ 方程组 } \begin{cases} -\frac{1}{2} = x^2 + y^2, \\ x + y = \frac{9}{2} \end{cases} \text{ 无解.}$$

$$\text{因此 } x = y, \text{ 故 } \begin{cases} \frac{\partial L}{\partial \lambda} = 2x^2 - z = 0, \\ \frac{\partial L}{\partial \mu} = 2x + z - 4 = 0 \end{cases}, \text{ 解得 } \begin{cases} x = y = 1, \\ z = 2. \end{cases} \text{ 或 } \begin{cases} x = y = -2, \\ z = 8. \end{cases}$$

比较可知距离原点最近的点为  $(1, 1, 2)$ .

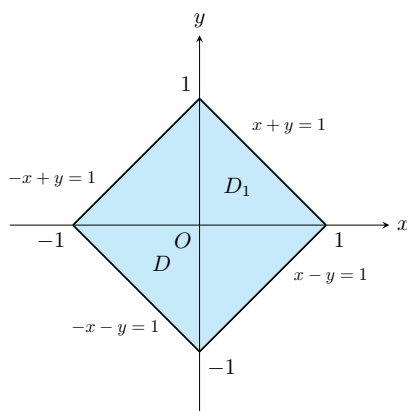
### 7. Solution.

积分区域如图所示, 其中  $D_1 = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$ .

由于  $D$  关于  $y$  轴对称, 函数  $y = x \sin(x^2 + y^2)$  关于  $x$  是奇函数,

所以  $\iint_D x \sin(x^2 + y^2) dx dy = 0$ . 再由对称性可得

$$\begin{aligned} I &= \iint_D x^2 y^2 dx dy = 4 \iint_{D_1} x^2 y^2 dx dy \\ &= 4 \int_0^1 x^2 dx \int_0^{1-x} y^2 dy = \frac{4}{3} \int_0^1 x^2 (1-x)^3 dx \\ &= \frac{4}{3} \int_0^1 (1-x)^2 x^3 dx = \frac{4}{3} \int_0^1 (x^5 - 2x^4 + x^3) dx \\ &= \frac{4}{3} \left( \frac{1}{6} - \frac{2}{5} + \frac{1}{4} \right) = \frac{1}{45}. \end{aligned}$$

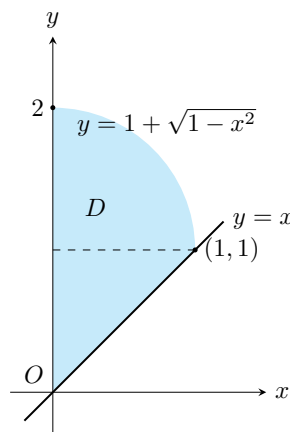


### 8. Solution.

积分区域如图所示.

用极坐标代换, 圆弧  $y = 1 + \sqrt{1 - x^2}$  对应的极坐标方程为  $r = 2 \sin \theta$ , 所以

$$\begin{aligned} I &= \iint_D xy dx dy = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^{2 \sin \theta} r^3 \cos \theta \sin \theta dr \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos \theta \sin \theta d\theta \int_0^{2 \sin \theta} r^3 dr = 4 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos \theta \sin^5 \theta d\theta \\ &= 4 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^5 \theta d \sin \theta = \frac{2}{3} \sin^6 \theta \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= \frac{2}{3} \left( 1 - \frac{1}{8} \right) = \frac{7}{12}. \end{aligned}$$



### 9. Solution.

交换积分次序,

$$\begin{aligned} I &= \int_0^1 dx \int_x^1 e^{-y^2} dy = \int_0^1 dy \int_0^y e^{-y^2} dx \\ &= \int_0^1 y e^{-y^2} dy = -\frac{1}{2} \int_0^1 e^{-y^2} d(-y^2) \\ &= -\frac{1}{2} e^{-y^2} \Big|_0^1 = \frac{1}{2} \left( 1 - \frac{1}{e} \right). \end{aligned}$$

10. **Solution.** 用球坐标代换, 锥面  $z = \sqrt{x^2 + y^2}$  对应的球坐标方程为  $\varphi = \frac{\pi}{4} (z \geq 0)$ , 所以

$$\begin{aligned} I &= \iiint_{\Omega} \frac{1}{\sqrt{x^2 + y^2 + z^2}} \mathrm{d}v = \int_0^{2\pi} \mathrm{d}\theta \int_0^{\frac{\pi}{4}} \mathrm{d}\varphi \int_0^{\frac{1}{\cos \varphi}} \frac{\rho^2 \sin \varphi}{\rho} \mathrm{d}\rho \\ &= 2\pi \int_0^{\frac{\pi}{4}} \sin \varphi \mathrm{d}\varphi \int_0^{\frac{1}{\cos \varphi}} \rho \mathrm{d}\rho = 2\pi \int_0^{\frac{\pi}{4}} \sin \varphi \cdot \frac{1}{2} \cdot \frac{1}{\cos^2 \varphi} \mathrm{d}\varphi \\ &= -\pi \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 \varphi} \mathrm{d}(\cos \varphi) = \pi \cdot \frac{1}{\cos \varphi} \Big|_0^{\frac{\pi}{4}} = \pi (\sqrt{2} - 1). \end{aligned}$$

## 2 综合题 (每小题 8 分, 共 40 分)

11. **Solution.**  $\begin{cases} u = x - 2\sqrt{y}, \\ v = x + 2\sqrt{y} \end{cases}$  两边全微分, 得  $\begin{cases} \mathrm{d}u = \mathrm{d}x - \frac{1}{\sqrt{y}} \mathrm{d}y, \\ \mathrm{d}v = \mathrm{d}x + \frac{1}{\sqrt{y}} \mathrm{d}y \end{cases}$ ,

所以  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 1$ ,  $\frac{\partial u}{\partial y} = -\frac{1}{\sqrt{y}}$ ,  $\frac{\partial v}{\partial y} = \frac{1}{\sqrt{y}}$ .

由链式法则,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v},$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = -\frac{1}{\sqrt{y}} \cdot \frac{\partial z}{\partial u} + \frac{1}{\sqrt{y}} \cdot \frac{\partial z}{\partial v},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) = \frac{\partial^2 z}{\partial u^2} \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial u \partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v \partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2} \cdot \frac{\partial v}{\partial x} = \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2},$$

$$\begin{aligned} \frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y} \left( -\frac{1}{\sqrt{y}} \cdot \frac{\partial z}{\partial u} + \frac{1}{\sqrt{y}} \cdot \frac{\partial z}{\partial v} \right) \\ &= \frac{1}{2y\sqrt{y}} \cdot \frac{\partial z}{\partial u} - \frac{1}{\sqrt{y}} \left( \frac{\partial^2 z}{\partial u^2} \cdot \frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u \partial v} \cdot \frac{\partial v}{\partial y} \right) - \frac{1}{2y\sqrt{y}} \cdot \frac{\partial z}{\partial v} + \frac{1}{\sqrt{y}} \left( \frac{\partial^2 z}{\partial v \partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v^2} \cdot \frac{\partial v}{\partial y} \right) \\ &= \frac{1}{2y\sqrt{y}} \cdot \left( \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) + \frac{1}{y} \left( \frac{\partial^2 z}{\partial u^2} - 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right). \end{aligned}$$

所以  $\frac{\partial^2 z}{\partial x^2} - y \frac{\partial^2 z}{\partial y^2} - \frac{1}{2} \frac{\partial z}{\partial y} =$

$$\begin{aligned} &\left( \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right) - y \left[ \frac{1}{2y\sqrt{y}} \cdot \left( \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) + \frac{1}{y} \left( \frac{\partial^2 z}{\partial u^2} - 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right) \right] \\ &- \frac{1}{2} \left[ -\frac{1}{\sqrt{y}} \cdot \frac{\partial z}{\partial u} + \frac{1}{\sqrt{y}} \cdot \frac{\partial z}{\partial v} \right] \\ &= \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{1}{2\sqrt{y}} \cdot \left( \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) - \left( \frac{\partial^2 z}{\partial u^2} - 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right) + \frac{1}{2\sqrt{y}} \cdot \left( \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \\ &= 4 \frac{\partial^2 z}{\partial u \partial v}. \end{aligned}$$

因此  $\frac{\partial^2 z}{\partial x^2} - y \frac{\partial^2 z}{\partial y^2} - \frac{1}{2} \frac{\partial z}{\partial y} = 0 \Leftrightarrow \frac{\partial^2 z}{\partial u \partial v} = 0$ .



12. **Solution.** 当  $x \rightarrow 0, y \rightarrow 0$  时,  $\frac{x^2 y^2}{(x^2 + y^2)^{\frac{3}{2}}} \leq x^2 \sqrt{|y|}$ , 而  $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} x^2 \sqrt{|y|} = 0$ ,

所以由夹逼定理可知  $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^2 y^2}{(x^2 + y^2)^{\frac{3}{2}}} = 0 = f(0, 0)$ , 函数  $f(x, y)$  在原点处连续.

又  $f(x, 0) = f(0, y) \equiv 0$ , 所以  $f_x(0, 0) = f_y(0, 0) = 0$ , 因此函数  $f(x, y)$  在原点处的偏导数存在.

考察

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{f(x, y) - f(0, 0) - f_x(0, 0)x - f_y(0, 0)y}{\sqrt{x^2 + y^2}} = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^2 y^2}{(x^2 + y^2)^2}.$$

当  $y = x, x \rightarrow 0, y \rightarrow 0$  时,

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^2 y^2}{(x^2 + y^2)^2} = \lim_{x \rightarrow 0} \frac{x^4}{(2x^2)^2} = \frac{1}{4},$$

所以  $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^2 y^2}{(x^2 + y^2)^2} \neq 0$ , 因此函数  $f(x, y)$  在原点处不可微.

13. **Solution.**  $\nabla F|_M = (ay^2 + 3cx^2z^2, 2axy + bz, 2cx^3z + by)|_{(1, 2, -1)} = (4a + 3c, 4a - b, 2b - 2c)$ ,

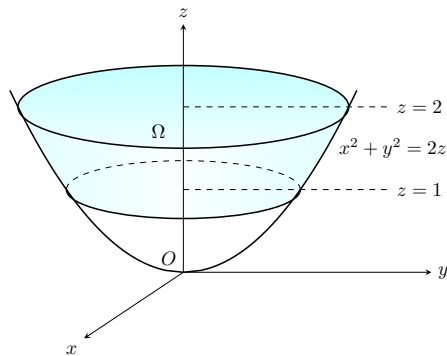
$$\text{由题意可知 } \nabla F|_M // (1, 0, 0), \text{ 所以 } \begin{cases} 4a - b = 0, \\ 2b - 2c = 0, \\ 4a + 3c = 64 \end{cases}, \text{ 解得 } \begin{cases} a = 4, \\ b = 16, \\ c = 16. \end{cases}$$

14. **Solution.**

由题可知旋转曲面的方程为  $x^2 + y^2 = 2z$ , 积分区域  $\Omega$  如图所示.

用截面法,

$$\begin{aligned} I &= \iiint_{\Omega} z \, dv = \int_1^2 dz \iint_{x^2 + y^2 \leq 2z} z \, dx \, dy \\ &= 2\pi \int_1^2 z^2 \, dz \\ &= \frac{2\pi}{3} (8 - 1) = \frac{14\pi}{3}. \end{aligned}$$



15. **Solution.** 如图所示, 联立  $\begin{cases} x^2 + y^2 + z = 4, \\ x^2 + y^2 + z^2 = 4z \end{cases}$  得  $4 - z + z^2 = 4z$ ,

解得  $z = 1$  或  $z = 4$ , 因此抛物面和球面的交线为  $x^2 + y^2 = 3$ .

抛物面  $x^2 + y^2 + z = 4$  将球体  $x^2 + y^2 + z^2 \leq 4z$  分为  $\Omega_1$  和  $\Omega_2$  两部分.

用截面法,

$$\begin{aligned}
 V_2 &= \iiint_{\Omega_2} dv = \int_0^1 dz \iint_{x^2+y^2 \leq 4z-z^2} dx dy + \int_1^4 dz \iint_{x^2+y^2 \leq 4-z} dx dy \\
 &= \pi \int_0^1 (4z - z^2) dz + \pi \int_1^4 (4 - z) dz \\
 &= \pi \left[ \left( 2z^2 - \frac{z^3}{3} \right) \Big|_0^1 + \left( 4z - \frac{z^2}{2} \right) \Big|_1^4 \right] \\
 &= \pi \left[ \left( 2 - \frac{1}{3} \right) + \left( 16 - 8 - 4 + \frac{1}{2} \right) \right] \\
 &= \pi \left( \frac{5}{3} + \frac{9}{2} \right) = \frac{37\pi}{6}.
 \end{aligned}$$

故  $V_1 = \frac{4\pi}{3} \cdot 2^3 - V_2 = \frac{32\pi}{3} - \frac{37\pi}{6} = \frac{27\pi}{6}$ , 所以体积比  $\frac{V_1}{V_2} = \frac{27}{37}$ .

