Problem Chosen	2020	Team Control Number
B	MCM/ICM Summary Sheet	**2013836**

Analytical Shape Design For The Most Durable Sandcastle Foundation

Summary

Sandcastle building, a recreational activity popular among beach goers, offer people with opportunities to create their exclusive sand-made works and making a sandcastle foundations is typically the first step of simple or complicated sandcastles. However, sandcastle foundations are always threatened by external destructive power. Since that not all sandcastles preform the same in dangerous surroundings, people may wonder what should be done to create a more durable sandcastle foundation and whether there is a desired shape of it.

In this paper, a cellular automata based mathematical model is established to find the answer. Firstly, we derived models for tides and sea waves by analyzing the movement of seawater. Accordingly, the internal force analysis and external force analysis were carried out, the former includes viscous force model and collapse model, the latter is comprised of impact force and lift. Next, we demonstrated the process of force analysis after dividing the relative position of sand cells into different cases.

Furthermore, sand-to-water proportion influences the stability of sandcastle foundation by changing the value of viscous force, which is the basis for identifying the optimal sand-to-water proportion. Afterwards we extend our model with consideration of rain, whose impact force is much more larger than that of seawater. Likewise, sand-to-water mixture proportion also changes when surface contacts water.

In computer simulation, we generated initial sandcastle foundations on their 2-D embryos with different gradients. Through iteration algorithm, a series of best 3-dimensional geometric shapes are found, featured with shell-shaped cross section. The ideal shape can be divided into three parts. Part A buffers the incoming sea water and decelerates the water without being broken. Part B cuts off the seawater flow directly, while it could further mitigate the seawater. Part C is the back bulk whose construction is steady, containing most of the sand. In parallel, we figured out the optimal sand-to-water mixture proportion, which is 2.63, or 2.70 in the rain. In the end, we listed practical strategies to make the sandcastle foundation last longer: the adjustment of building time, adding supporting structures and improving building materials.

Finally, we finished sensitivity analysis and summarized the strengths and weaknesses of the model.

Keywords: Cellular automata, Evolution of three-dimensional complex space-time systems, The longest lasting sandcastle foundation

关注数学模型
获取更多资讯

Contents

关注数学模型

获取更多资讯

关注数学模型
获取更多资讯

1 Introduction

1.1 Problem Background

Building sandcastle is both an attractive and relaxing recreational activity to do on the seashore. Through imagination and hands-on practice, one can build sandcastles from small mounds of sand to complicated works that have similar shapes of real castles. Sandcastles creation has been very popular among children and adults form a long time globally. An typical and common method to create sandcastle begins with a foundation, usually a nondescript mound of wetted sand. Based on the foundation, beach goers further reshape, modify and decorate their work by their own hands.

Figure 1: Hand drawing of sandcastle

Unfortunately, sandcastles are vulnerable because of rain and ocean waves accompanied with rising tides. What draws people's interest is that sandcastles react differently to erosion from external environment.So, it is valuable to get an optimal design for sandcastle foundation.

1.2 Restatement of Problem

Not all sandcastles last for same time, even if they share same size and locate roughly the same distance from the water on the same beach. As a result, people try to find a best 3-dimensional geometric shape for sandcastle foundations. According to the given conditions, following issues are required to be discussed in this paper.

- A mathematical model identifying the best 3-dimensional geometric shape for sandcastle foundation under same circumstance. Identical building material should be used as well.
- Optimal sand-to-water mixture proportion for the sandcastle foundation without assistance including additives and materials.
- How the best shape design for sandcastle foundation affected by rain.
- Other strategies, if any, could be applied in order to improve the durability of sandcastle foundation.

关注数学模型
获取更多资讯

2 Assumptions and Justifications

The following basic assumptions are made to simplify problems.

- **Chemical erosion is negligible for sandcastle foundations.** Although state-of-the-art sandcastles can last for a long period, ordinary ones built by common beach visitors usually maintain their shape for several hours to a day. The case is that chemical erosion is not visible in quite a short time for chemical reactions.
- **Coast line near the sandcastle foundation is straight.** The diameter of sandcastle foundations are much smaller that the curvature radius of coast line in most occasions, which means that coast line can be treated as straight line.
- **The amount of rainfall keeps a constant at a time.** In reality, rainfall randomly changes, but to an area with similar size of a sandcastle, the rainfall will not fluctuate significantly in a few hours.
- **No impact caused by extreme accidents affect the sandcastle.** Our model focused the mechanical properties of the sandcastle foundation and its interaction with the water. Extreme accidents caused by human activities and extreme weather are out of our concern.

Other specific assumption, if is necessary, will be mentioned and illustrated while the model is building.

3 Symbols and Definitions

Partial symbols in this paper are displayed as shown in Table 1 and other symbols will be described later in the text.

Table 1: Nomenclature

Symbol	*Definition*
ρ	Sea water density
a	Length of a cubic cell
K	Sand-to-water mixture proportion
k	Water-to-sand mixture proportion
s	Gradient of the initial foundation
s'	Local gradient
s'_c	Critical local gradient
c	Velocity of sea wave
l	Length of sandcastle foundation
f	Viscous force
F_{lift}	Lift force
F_i	Impact force
v	Speed of sea water
η	Coefficient of viscosity
R	Residual rate

关注数学模型
获取更多资讯

4 Model Building

4.1 Cellular Automata Sandcastle Foundation Model

4.1.1 Cell

Cellular automata, or CA, is a well-known mathematic model who has already found applications in many areas such as computer science, physics and traffic engineering. Usually, cellular are discrete in time, state and space. Unlike general dynamical models, cellular automata is determined by a series rules constructing the model instead of strict physical equations.

The smallest unit in cellular automata, namely cell, represents a spacial cube whose position and volume are constants. Increasingly, whole space a sandcastle foundation occupies is divided into a host of cubes next to each other regularly. Each cell has its own parameters whose value will change in every time step. Table2 lists most of the parameters of a cell.

For the purpose of simplification, height and distance are also counted in the length of cubic cell. If the length of cubic is$a = 5mm$, $d = 2cm$ can be rewrite as $d = 4$, et cetera.

4.1.2 Build sequcence of the initial sandcastle foundation

A typical sandcastle is established on its foundation, while all the sandcastle foundations are made of nothing but a large amount of wet sand. With one's owm hands and simple handy tool, sandcastle must be created in a reasonable manner. Likewise, sandcastle foundation is responsible to hold complicated sandcastle structure, indicating that it has be be firm enough. Grains of sand or sand blocks are forbidden to be hung in the air with nothing solid supporting. Therefore, one layer of cubic cells in sandcastle foundation must be not larger than the layer below in area. The creation of sandcastle foundation, based on this principle, is able to conduct in layers from bottom to top. For convenience, the lowest layer of cells in the sandcastle foundation is called embryo.

None can tell which kind of 3-dimensional geometric shape is the best among all the possible choices theoretically, neither the best embryo, before the entire model is solved. Practically, embryo can be daily seen shapes like square, triangular and circle. To show the approach to forming the initial sandcastle, take square as an example. After generating the embryo, the next layer of cells is able to be generated according

Table 2: Cell parameters

Symbol	*Meanings*
X	x-axis coordinate (perpendicular to coast line)
Y	y-axis coordinate (parallel to coast line)
Z	z-axis coordinate (vertical to the beach)
K	Sand-to-water mixture proportion
k	Water-to-sand mixture proportion
s'	Local gradient
s'_c	Critical local gradient

关注数学模型
获取更多资讯

to a certain *gradient* s.

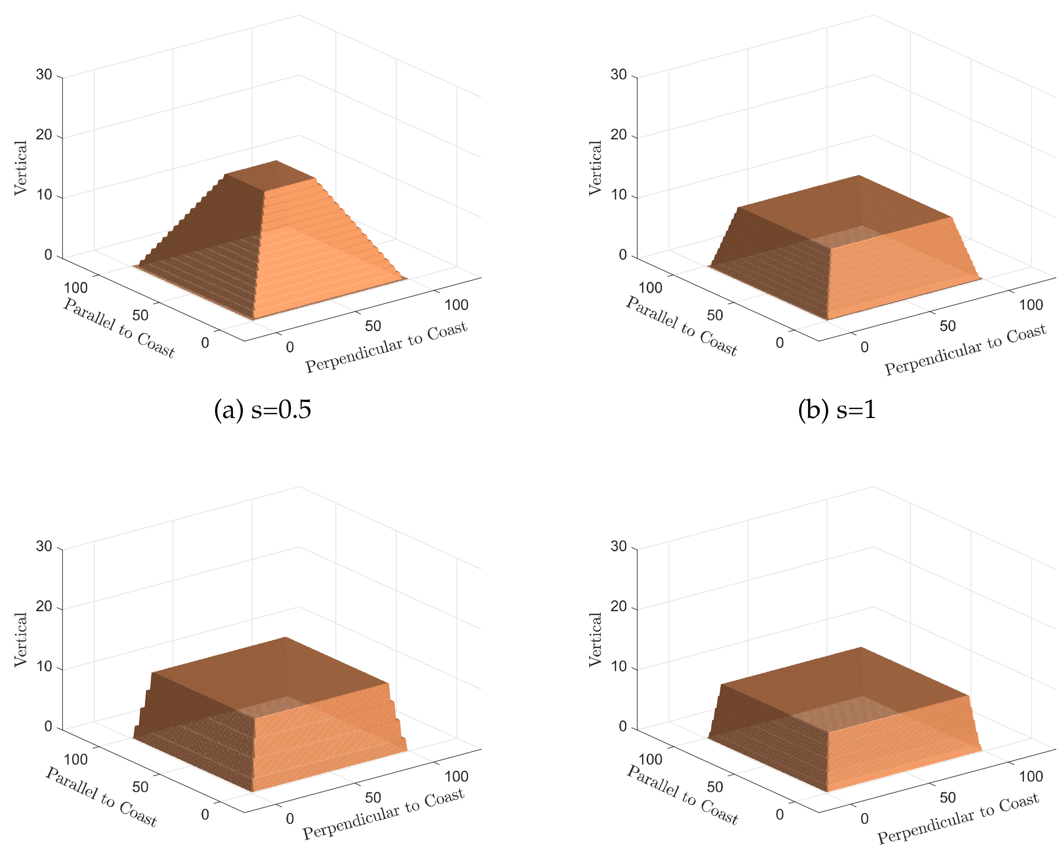
There are three steps to generate a new layer based on the lower layer next to it until the sand amount limitation is reached.

1. Virtually remove the edge whose width is the shrinkage w and get a new square R'.
2. Put the new square R' on the original layer as the next layer.
3. Repeat the second step for n times.

Gradient is defined as the ratio of n and w, namely

$$s = \frac{n}{w}. \tag{1}$$

As we can see, gradient expresses the extent of inclination of the sandcastle castle foundation. In this manner, one can figure out the whole shape of the initial sandcastle foundation. Figure 2 displays a group of initial sandcastle foundations generated by different shrinkage. Note that they are roughly same in volume and share the identical square-shape embryo.



(a) s=0.5 (b) s=1

(c) s=1.5 (d) s=2

Figure 2: A group of initial sandcastle foundations generated by different gradient

关注数学模型
获取更多资讯

4.2 Tides and sea waves

Wave and tide are familiar phenomena one can observe on the seashore. Generally speaking, waves refer to fluctuations of sea water triggered by the force of wind, while tides are caused by gravity of the moon. Result of scientific research indicates that tide often happens approximately twice per day, with more energy attached, and waves are much more frequent and milder [1].

4.2.1 Tides

In consequence, variation of sea level brought by tides are able to be fitted through a series of sinusoidal functions featured with different frequency and magnitude. In addition, the maximum of the fitted function should be less than 24 hours to satisfy the periodicity. Aiming at quantify the behavior of tides, following parameters is specified.

The figure below determined how tides influence the height of sea surface. Figure 3 is the plot of $y_t(t)$.

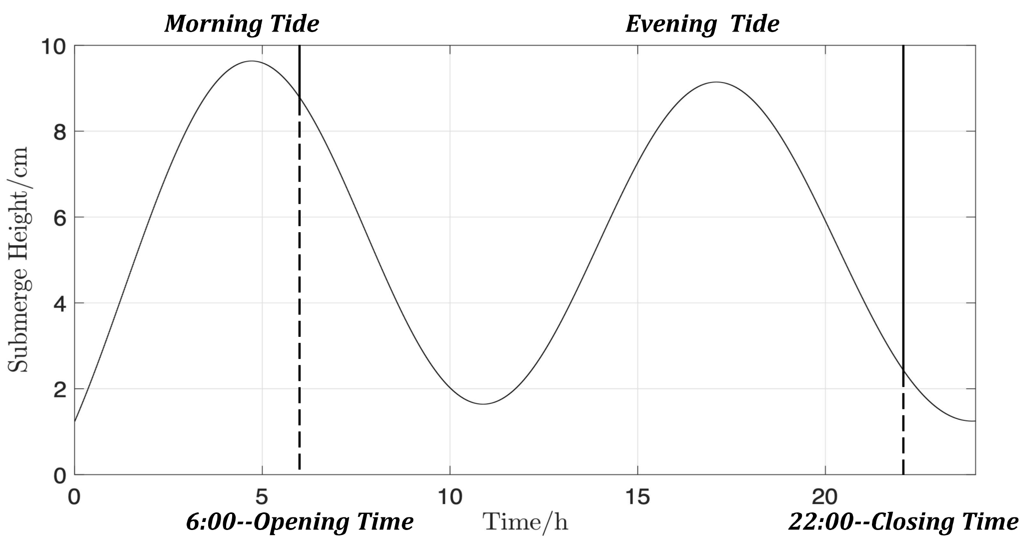


Figure 3: Tide

Morning tide and evening tide has been labeled. To be mentioned, beaches for recreation are not 24-hour accessible. They open at 6:00 and close at 22:00.

4.2.2 Sea waves

As it has been mentioned earlier in this section, waves are much more frequent and milder compared with tides. The velocity of wave in shallow water $c$can be calculated by

$$c = \sqrt{h \cdot g}. \tag{2}$$

In Equation 2, g is gravitational acceleration and h is the height of the sea wave top. SandBeach often has a very small dig angleθ, for example 5°. The dig angle θ causes a speed gap when the rising top of rising wave arrives and leaves the sandcastle, if its has enough energy to overflow the entire bottom of the sandcastle foundation. The mechanism is similar when the wave comes back.

关注数学模型
获取更多资讯

Arriving speed v_1 and leaving speed v_2 is computed through Equation 2.

$$v1 = \sqrt{gh_1}v2 = \sqrt{gh_2} \tag{3}$$

Then, let h be the depth of seawater,

$$h_1 = h \tag{4}$$
$$h_2 = h - l\sin\theta \tag{5}$$

where l is the length of sandcastle foundation in y-axis.

$$v_1 = \sqrt{gh} \tag{6}$$
$$v_2 = \sqrt{g(h - l\sin\theta)} \tag{7}$$

In oder to further simplify the model, we use the average speed to replace the successive varying speed. Since the front side and back side of sandcastle foundation receive unequal impact force of seawater, calculation of average speed is separated into front side $\bar{v_1}$ and back side $\bar{v_2}$. Assume the acceleration and deceleration of seawater speed stays at a constant value, overall average speed $\bar{v}$ is

$$\bar{v} = \frac{\sqrt{v_1^2 + v_2^2}}{2} \tag{8}$$

and

$$\bar{v_1} = \frac{v_1 + \bar{v}}{2} \tag{9}$$
$$\bar{v_2} = \frac{v_2 + \bar{v}}{2} \tag{10}$$

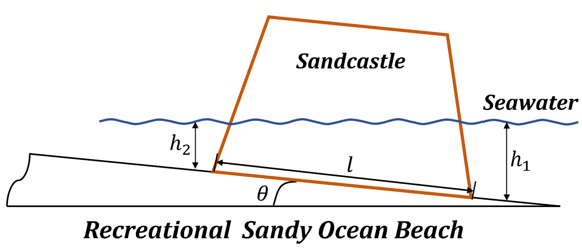


Figure 4: The raise of seawater

Besides, energy attached to sea wave is concentrated in seconds, so that we set the period of sea wave to T_w, which is to say the sea wave comes every T_w, called wave period.

4.3 Internal force analysis

4.3.1 Viscous force

The reason why a mound of sand or sandcastle foundations are able to endure external force, to some extent, is viscous force f between grains of sand. In our model,

关注数学模型
获取更多资讯

viscous force exists between two tightly connected cells, hindering their relative motion. Additionally, viscous depends on the sand-to-water mixture proportion K.

Viscous force is a complex of several microcosmic interactions, including friction, electrostatic force and so on[5][6]. In a general way, absolute dry sand grains stick to each other by friction. As the sand-to-water mixture decreases, growing amount of water leads to the strengthening electrostatic force due to the polarity of H_2O molecule, noticing that grains of sand possess very limited polarity. Nonetheless, too much water could impair the friction because its liquidity.

Consequently, there is always a trade off between the liquidity and electrostatic force, enhancing and weakening viscous force respectively. Moreover, we tend to admit that there is a certain water-to-sand mixture proportion $\bar{k}$ which leads to the peak of viscous force. In our model, this relationship is fitted by an exponential piecewise function.

$$f = \begin{cases} f_0 \cdot Ae^{-\frac{(k-\bar{k})^2}{k_1}} \\ f_0 \cdot Ae^{-\frac{(k-\bar{k})^2}{k_2}} \end{cases} \tag{11}$$

In the expression of viscous force f, f_0 is the viscous force when the sand cell is completely dry k_1 and k_2 is involved to adjust the speed of variation of water-to-sand proportion is greater or less than $\bar{k}$. We also employ A to show how much the viscous force gets bigger after being mixed with water. It should be mentioned that k is water-to-sand mixture proportion, the reciprocal of sand-to-water proportion k.Related research suggested that the stickiest wet is obtained when the amount of substance of sand and water is $1:1$.Therefore, their volume ratio is $\bar{k} = 0.445$.

Relation between viscous force and water-to-sand proportion is sketched in Figure 5.

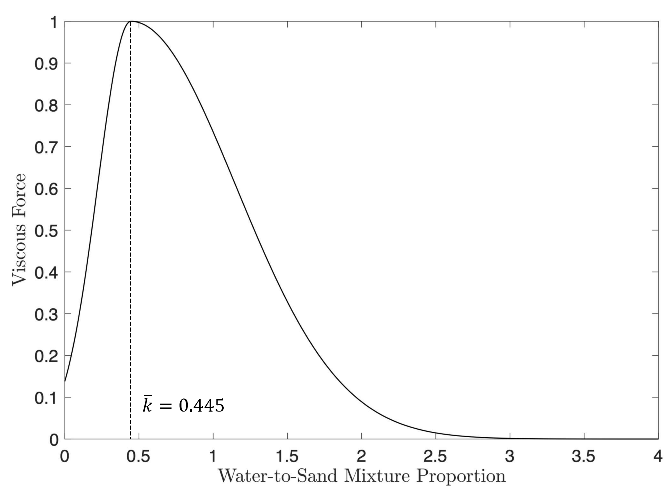


Figure 5: Relation between viscous force and water-to-sand proportion

4.3.2 Slide and collapse

After modeling viscous force, it is ready for us to do stability analysis about sandcastle foundations consisting of a host of cells. While the shape of sandcastle foundation is changing continuously, so that it is of great necessity to figure out wether the structure

关注数学模型
获取更多资讯

is stable enough to hold itself. Sandpile model[2], a quantitative clarity of SOC (self-organized criticality), is suitable to calculate the stability of a set of particles.

Firstly, let us introduce the concept of local gradient s'. At any instant, one can find the highest cell among all the cells whose x-ordinate and y-ordinate is (X, Y). The z-ordinate of the highest cell, denoted by H, therefore, is a function of (X, Y), namely

$$H = H(X, Y)$$

Then, the local gradient is defined as

$$s'(X, Y) = max\{H(X, Y) - H(X \pm 1, Y \pm 1)\} \tag{12}$$

As the definition of local gradient s' indicates, it quantify the potential a cell might slides from the sandcastle foundation. P_s is the slide probability depends on s' and s'_c.

$$P_s = \begin{cases} 1 & s' > s'_c \\ 0 & s' < s'_c \\ P_0 & s' = s'_c \end{cases} \tag{13}$$

If $s' > s'_c$ where s'_c is the critical local gradient, the cell whose coordinate is (X, Y, H) is bound to slide away. Similarly, when s' is less than s'_c, the cell is stable and temporarily remains its position. At critical condition $s' = s'_c$, the cell slide with a probability P_0 possibly triggering other slides of other cells. It is even worse that the slide of one cells, may trigger the slides of others in the future, possibly leading to the collapse in part or the entire sandcastle foundation.

Critical local gradient s'_c is dependent on the normalized viscous force f_N. Their functional relationship is determined by Equation14. The stickier wet sand leads result in larger critical local gradient.

$$s'_c = \begin{cases} 1 & 0 \leq f_N < 0.2 \\ 2 & 0.2 \leq f_N < 0.4 \\ 3 & 0.4 \leq f_N < 0.6 \\ 4 & 0.6 \leq f_N < 0.8 \\ 5 & 0.8 \leq f_N \geq 1 \end{cases} \tag{14}$$

To summarize, static structure of sandcastle foundation is threatened by the following items after erosion of sea water and rain.

- Change of sandcastle foundation shape
- Change of water-to-sand mixture proportion k
- Impact of raindrop

关注数学模型
获取更多资讯

4.4 External force analysis

4.4.1 Impact force and scouring effect

Role of sea water in the erosion in the sandcastle foundation is divided into two effects, scouring and floatage. Scouring effect describes how forces perpendicular to the coastline break the sandcastle foundation. It is of importance that sea water may hit the sandcastle foundation from the front and back where rising of water is high enough to do so. Meanwhile, floatage, another effect of sea water, mainly interacting with the upper surface of the sandcastle foundation, will be explained later in this section.

How impact force exactly hit a specific cell depends on the relative position of the cell. Since the viscous force only occurs between closely contacted cells, it becomes necessary to divided different situations on the basis of cells surrounding a selected cell. Figure 6 displays all the situations in this model where the lower layers are hidden.

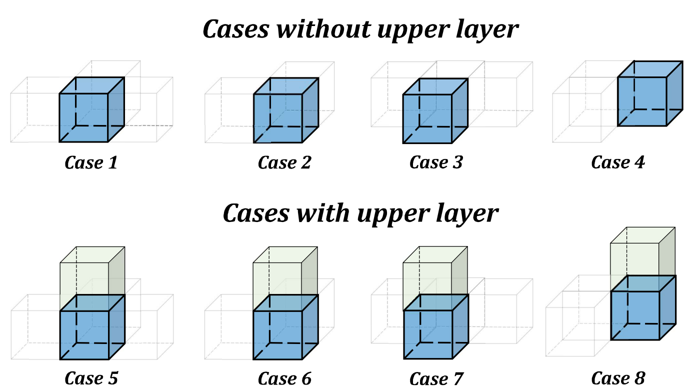


Figure 6: 8 cases of cell meeting impact force

Blue cubes cell are being studied in each cases. Studied cell in Cases 1-4 does not have any cell on the upward side, while they do in Cases 5-8. Next, Case 2 and Case 4 are chosen as example cases to show the course of force analysis.

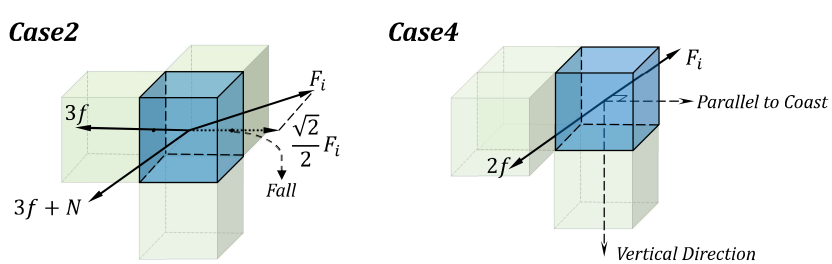


Figure 7: Force analysis of Case 2 and Case 4

In case 2, the direction of impact force $\vec{F_i}$ is parallel to the diagonal of the upper side of the studied objective cell, along with the pass of seawater flow. As is marked in Figure 7, $\vec{F_i}$ is resolved into the rightward direction and inward direction. Actually, only the resolved force in rightward direction, whose value is $\frac{\sqrt{2}}{2}F_i$ could drag the cell away. Another resolved force can not make an effect because cells behind the

关注数学模型
获取更多资讯

studied cell can provide extremely large supporting force N. Simultaneously, three surrounding cells causes three times of viscous forces, aiming at preventing the blue cell from leaving. Accordingly, if the inequation

$$\frac{\sqrt{2}}{2}F_i > 3f$$

stands, blue cell will be scoured apart from the sandcastle foundation, vice versa.

In the same manner, conditions of scouring can be obtained. Because there are no cell behind the studied blue cell, no supporting force might work as a barrier to stop the incoming seawater. Neither exists the rationality of resolving impact force. As a consequence, scouring condition of Case 4 is

$$F_i > 2f$$

Scouring conditions of all cases are demonstrated in Table 3.

Up to now, how the impact force react on each cells has been explained in detail, but question about exact value of impact force still remains. Fortunately, the Stokes Law can provide us a realistic approach. The content of Stokes Law is: a spherical object in viscous fluid receives a resistance whose value is

$$F_{i0} = 3\pi\eta\mu D \tag{15}$$

Viscous coefficient, relative velocity are denoted by η and μ, while the diameter of spherical object is D. A grain of sand, regarded as a sphere, endure resistance whose value can be derived through Equation 15 due to its relative movement with the seawater. So does this principle establishes when a sand cell is being discussed. Impact force F_i is the sum of resistance all the sand grains bear in a cell. Finally, impact force F_i has the form of

$$F_i = \gamma\mu \tag{16}$$

In the expression above, γ is a ratio coefficients, a combination of factors after summation.

Table 3: Scouring conditions of all cases

Case Number	*Scouring condition*
1	$F_i > 3f + N$
2	$\frac{\sqrt{2}}{2}F_i > 3f$
3	$F_i > f + N$
4	$F_i > 2f$
5	$F_i > 5f + N$
6	$\frac{\sqrt{2}}{2}F_i > 4f$
7	$F_i > 3f + N$
8	$F > 3f$

关注数学模型
获取更多资讯

4.4.2 Lift and floatage effect

Apart from scouring effect, upper surface cells face risk of floating due to the lift force. Bernoulli's principle, a well-known theorem in fluid mechanics, explains why a number of sand might be found on the surface of water even if the density of sand grain is greater than that of sea water[4]. Equation 17 is one of expressions of Bernoulli's principle.

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_{01} = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_{02} \tag{17}$$

In Equation17, P_1,P_2 are the intensities of pressure at the top and bottom of a cell. Besides, h_{01},h_{02} represent the height of the top and bottom of the cell. rho is the density of sea water. As for a single cell in the sandcastle foundation, $v_2 = 0$ because there is no fluid under the cell, resulting

$$P_2 - P_1 = \frac{1}{2}\rho v_1^2 + \rho \Delta h_0 \tag{18}$$

Then, applying Equation 18 to a single cell in a sandcastle foundation.

$$\Delta h_0 = a \tag{19}$$
$$S = a^2 \tag{20}$$
$$V = a^3 \tag{21}$$
$$F_{lift} = (P_2 - P_1) \cdot S \tag{22}$$
$$\mu = v_1 \tag{23}$$

In Equation 19, S is the area of each side of cubic cell, and V is the volume of the cell. One can obtain the function of lift as follow.

$$F_{lift} = \frac{1}{2}\rho \mu^2 S + \rho g V \tag{24}$$

Similarly, floatage effect can also be divided to cases. Five cases in Figure circumstance of a cell whose upper side is exposed to the air.

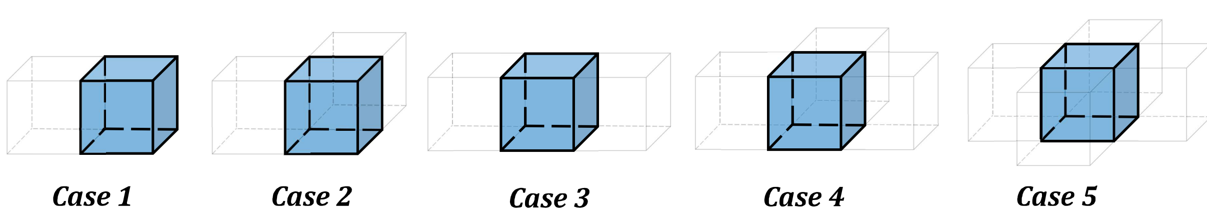


Figure 8: 5 cases of cells meeting lift force

For instance, in case 2, there are 3 cells surrounding the blue cell which is the studied sell. The blue has a tendency of flotage, while the gravity and viscous force is trying to keep it at the current position. Hence, the flotage condition is

$$F_{lift} > G + 3f$$

关注数学模型
获取更多资讯

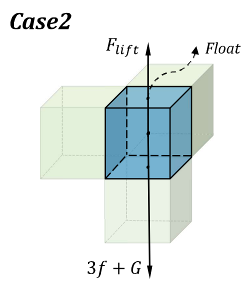


Figure 9: Force analysis of case 2

Floatage condition in all cases is also demonstrated in Table 4

Table 4: Floatage conditions of all cases

Case Number	*Floatage condition*
1	$F_{lift} > 2f + G$
2	$F_{lift} > 3f + G$
3	$F_{lift} > 3f + G$
4	$F_{lift} > 4f + G$
5	$F_{lift} > 5f + G$

4.5 Raining

During rainy days, sandcastle foundation is possible to be broken by raindrops, even if the sandcastle is far from the coast line. Raining affect the previous model from two aspects:

- Rainwater brings more water to the sandcastle foundation, changing the water-to-sand mixture proportion
- Raindrops impact the sandcastle foundation surface

Study illustrates that the impact of raindrop is much more stronger than the gravity and viscous force between sand cells. The impact of raindrop is so big that distinguishes the effect of rain from that of sea water. Hence, it is necessary to build a novel model for the rain.

A major character of the rain is that raindrop only falls on cells whose upper side is not holding another cell. Those cells can be divided into two categories.

I Only the upper side is exposed to the air.

II At least two sides of the cell is exposed to the air.

Cells belong to category I, surrounded by fives cells, is not able to be washed away by the raindrop. To the contrary, their water-to-sand mixture proportion will increase after the falling of raindrops. Cells in category II, on the other hand, will be washed away immediately be impacted by a raindrop. In the following part of the paper, we use λ as the relative rainfall, the ratio of real rainfall to the average rainfall in eastern US.

关注数学模型
获取更多资讯

4.6 Change of water-to-sand proportion

4.6.1 Seawater

Every time when waves hit the sandcastle foundations, water-to-sand ratio of the hit cell changes. The amount of change Δk negatively correlates with the current water-to-sand ratio k. Since $k > 0$, we use and negative exponential function to substitute the relationship.

$$\Delta k = \alpha e^{-k} \tag{25}$$

4.6.2 Raining

According to the raining model, rainwater sufficiently contacts with the cells belonging to category I. Thus, the variation of water-to-sand ratio Δk is independent from the current water-to-sand proportion, which is to say Δk is uniform while time goes.

$$\Delta k \propto \beta\lambda \tag{26}$$

where β is a constant representing how much rainwater is absorbed by the sandcastle foundation.

5 Model Validating

5.1 Residual ratio

The primary step of solving is to define the criteria of durability of a cell. *Residual ratio* R is involved to help us make judgments on the quality of sandcastle foundations. Residual ratio R is defined through

$$R = \frac{Amount\ of\ left\ cells\ after\ erosion\ n}{Initial\ amount\ of\ cells\ n_0} \tag{27}$$

It might be a method to manually define a threshold: sandcastle foundation whose residual ratio is lower than the threshold is not worthy any more, but the manually deciding threshold could be both difficult and unscientific. Wondering the longest lasting sandcastle foundation, a wiser way is to compare the residual ratio R within a same length of time.

5.2 The best 3-dimensional geometric shape of sandcastle foundation

The first requirement focused on the influence of tides and sea waves. With the insurance of approximately same amount of cells in sandcastle foundations, simulate how tides and sea water erode the sandcastle foundation whose embryo is elliptical. To have a good guarantee of the stability of the initial sandcastle foundation, sand-to-water mixture proportion is set to $k_0 = 0.445$ at beginning to maximize the viscous force. When submerge height is 6cm, relationship between residual ratio R and simulation time for 3 different gradients s is illustrated in Figure 10. The simulation process lasts for 2 hours. As Figure 10 tells, residual curves differ with each other under different gradients s. Residual ratio R of 0.5-gradient sandcastle foundation lie in bottom at the early beginning, but it exceeds others later. So, merely consider the erosion of

关注数学模型
获取更多资讯

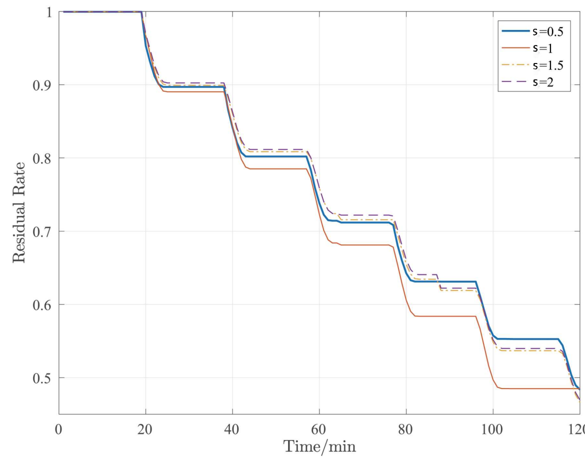


Figure 10: Relationship between residual ratios and different gradients

tides and sea waves, one ought to build the least gradient, namely milder ones legally, which coincides with common sense of life.

Meanwhile, residual ratio curves present same variation trends like a ladder. The ladder sphere suggested that losing of sand is segmented. In the steep part of the curve, sand loses dynamically while it loses slowly, nearly remaining the same amount in the slight part of the curves. This phenomenon, named after "*ladder effect*", indicates that in the platforms of the curves, sandcastles foundation exists a relative stable shape capable to resist the impact force and lift force, concurrently remain its stability. Finally, on account of growing water-to-sand proportion k, weakened viscous force triggers the instant lost of sand.

In order to grasp the features of steady shape, we find out somewhat similarity after plotting the shape of sandcastle foundations with different gradients when the residual ratio stays at a platform. This similarity is demonstrated with an example: a gradient $s = 2$ sandcastle foundation being eroded for 90mins.

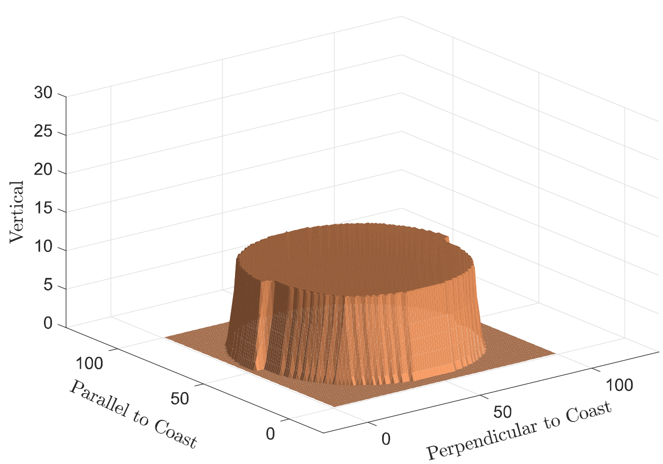


Figure 11: A stable sandcastle foundation in simulation

关注数学模型
获取更多资讯

In Figure 11, the front said of sandcastle foundation is blunt. As the sea water deepen, surface becomes sharper due to the scouring effect. Likewise, the shape of the front and back surface is unsymmetrical, divided by a bevel whose slope is roughly equal to the gradient, because seawater impact the front surface and the back surface to different extent.

Plenty times of simulation shows that sandcastle foundations generated by various shapes of embryos reach similarly state like the mentioned structure. In consequence, the best 3-dimensional geometric shape is obtained through iteration.

1. Generate an initial sandcastle foundation with arbitrary embryo
2. Simulate the sandcastle foundation for a fixed length of time
3. Make a new embryo geometrically similar to the cross section of the last foundation when it becomes stable
4. Repeat step 2 and 3 until obtaining an ideal 3-dimensional sandcastle model

The ideal 3-dimensional sandcastle foundation from two aspects is sketched in Figure12.

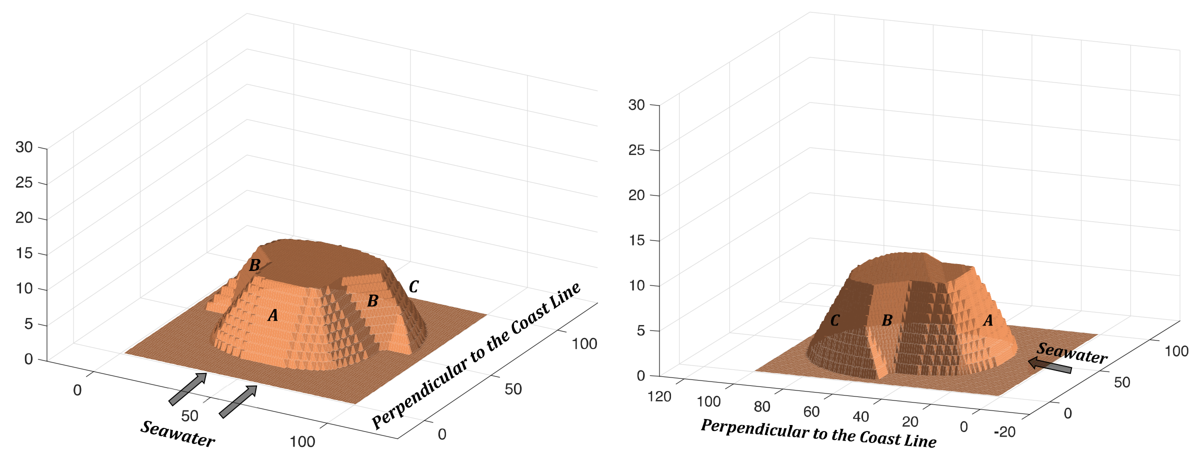


Figure 12: Ideal 3-dimensional sandcastle foundation model

This ideal 3-dimensional sandcastle foundation can be divided into 3 parts, marked as A,B and C. Part A buffer the incoming sea water and decelerate the water without being broken. Part B cuts off the seawater flow directly, while it could further mitigate the seawater. The back bulk is part C whose construction is steady, containing most of sand. All three parts should be large enough for cutting and reshaping.

5.3 Best sand-to-water mixture proportion

As is shown in Figure 13, In each curve, with water-to-sand mixture proportion increasing, the residual ratio increases first and then decreases.

Meanwhile, residual ratio curves present same variation trends like a ladder. This is due to the ladder effect. Although the ladder-shaped distribution makes the residual ratio corresponding to different water-to-sand mixture proportion roughly the same,

关注数学模型
获取更多资讯

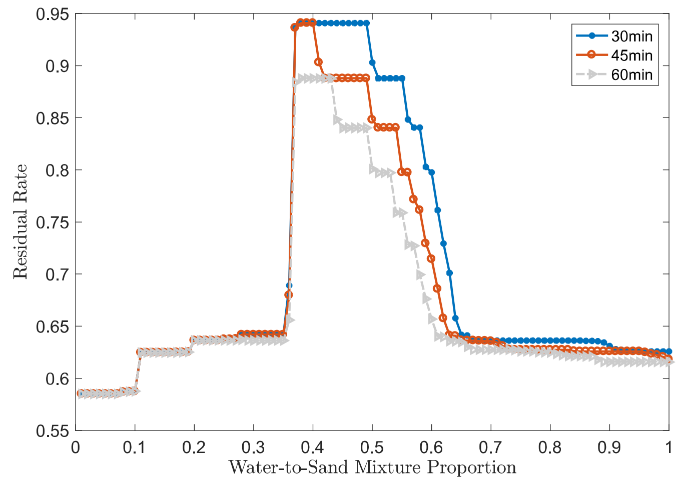


Figure 13: Solution of sand-two-water proportion

the optimal proportion of the three curves can still be determined from the data, which is the same—0.38.

Therefore, the optimal water-to-sand mixture proportion of the best sandcastle is

$$K = \frac{1}{k} = 2.63$$

which means that each liter of seawater is matched with 2.63 liters of sand to build the longest lasting sandcastle foundation.

5.4 Sandcastle foundation in the rain

Both seawater and rainwater is consider to identify best sandcastle foundation in the rain. Similar to the manner in Section 5.2, applying elliptical embryo to generate initial sandcastle foundation. Therein, water-to-sand proportion is set to 0.045 and relative rainfall is $\lambda = 1$. Simulation result of 30min is shown in Figure 14

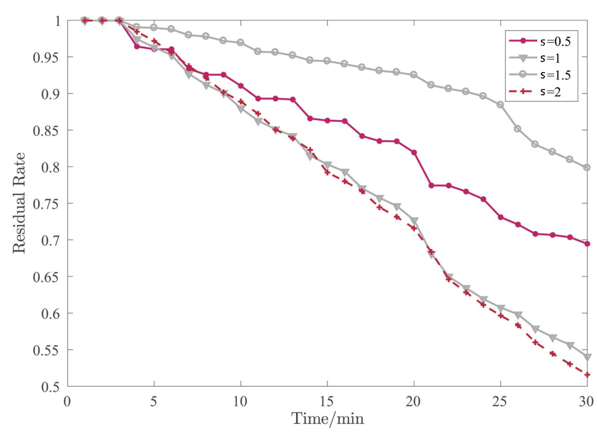


Figure 14: Residual ratio curves concerning rainwater

Residual ratio curves behave differently when the initial gradient varies. The curve marked by $s = 1.5$ is much more higher than others, which is to say this gradient pre-

关注数学模型
获取更多资讯

forms the best under combined action of rain and seawater. As a reason, $s = 1.5$ is a better value in this scenario. Moreover, under the erosion of rain, there is no "ladder effect" along the reducing of residual ratio R. Similar features appear in sandcastle foundations of other gradients. Figure 15 demonstrates a 30-minute simulation example where gradient $s = 1.5$.

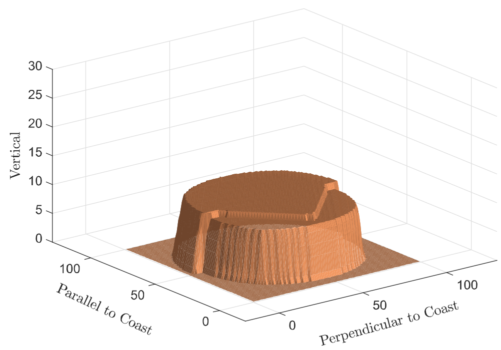


Figure 15: Simulation result of sandcastle foundation in the rain

The major differences between Figure 15 and Figure 11 is the existence of corner along the top. Sandcastle foundation faces erosion from the seawater and raindrops, causing dramatic shrinking.

After same way of iteration in Section 5.2, the sandcastle foundation is convergent to a stable shape in Figure 16.

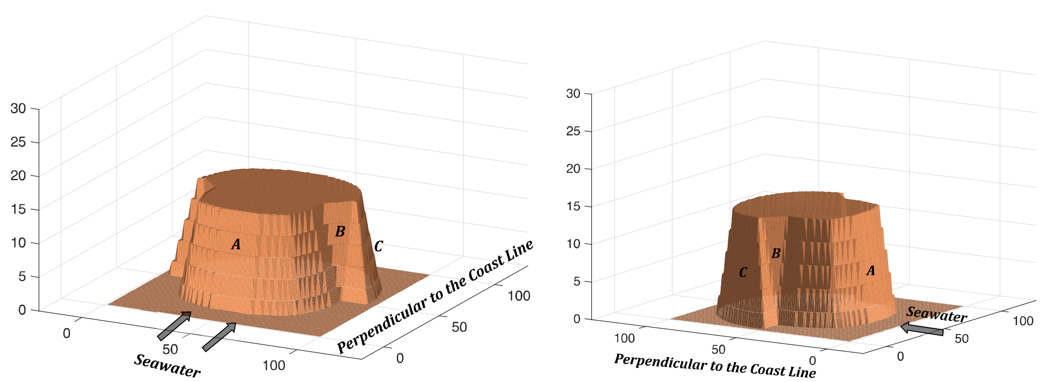


Figure 16: Ideal 3-dimensional sandcastle foundation in the rain

The shape in Figure 15 is still capable to block the seawater flow thanks to part A and B, and contain part C to store sand. However, the slope is steeper, decreasing the impact of raindrop. Simulation outcomes of the ideal sandcastle model in the rain shows that residual ratio $R = 0.91$ after 30 minutes, but sandcastle foundation of same volume but generated by the basic geometric shape embryo present much more lower residual ratio. All their residual ratio when after 30 minutes is lower than 0.8. In some occasions residual ratio R nearly reached 0.5.

To sum up, sandcastle foundation in Figure 16 is the optimal best 3-dimensional geometric shape.

关注数学模型
获取更多资讯

5.5 Best sand-to-water mixture proportion in rainy condition

After identifying the best 3-dimensional geometric shape of sandcastle in rainsimulate how water-to-sand proportion changes sandcastle. Residual ratio curve does not

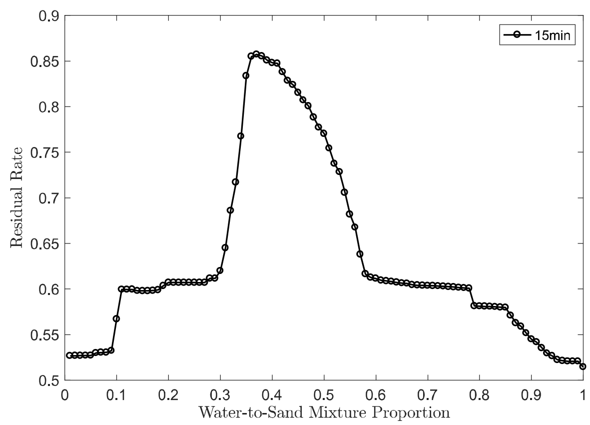


Figure 17: Relation between residual ratio and water-to-sand proportion in the rain

have a staircase shape, so it is easy to identify the best water-to-sand mixture proportion in rain0.37 and K is determined at the same time.

$$K = 2.70$$

5.6 Others strategies to make sandcastle last longer

- **Adjust the time of building sandcastle.** In the wave model simulated in this paper, the height of the wave will peak at 5:00 in the morning and 17:00 in the afternoon. Since the higher the wave, the faster the erosion of the sandcastle, to avoid the erosion of the waves, visitors can choose to start the sandcastle after the peak of the wave from 9:00-10:00 in the morning. A few hours after , the average height of waves will remain about a third of its peak and there will be no significant growth. Because of impact of low waves of on the sand castle is not very obvious, sand castle will naturally last longer.

- **"Dress" the sandcastle foundation.** To keep the sand castle as long as possible, the sand castle should be protected from the sea, rain and wind, and visitors often adds some protection. Small sand sculptures usually be added wooden boards on its surface, and covered its top with plastic film, which not only fixes the structure of the sculpture, but also protects the surface from the erosion of the waves and rain. Large sand sculptures often paste a thin layer of cement on the surface to maintain the water-sand ratio, reduce internal water loss and prevent internal collapse.

- **Change the materials used to build the sculpture.** The only materials used to build the sculpture are water and sand, but the purity of sand, and the size of sand grains, the viscous coefficient of water, these physical properties changes affect the most important properties of the sandcastle –the viscous force f, the higher the purity, the finer the particle, the greater the viscosity coefficient η, the

关注数学模型
获取更多资讯

greater viscous force between the sand cells f, the more stable the structure, the more resistant to outside erosion, the longer it will last. So adding glue as a binder, or using thinner, purer sand, will prolong the sandcastle's duration.

6 Sensitivity analysis

It is mentioned in the hypothesis that the amount of rainfall keepsăaăconstantăatăaătime, but the amount of rainfall at a time is necessarily different or even far from each other during a week or a month. In order to know exactly how rainfall affects on the model, the sensitivity analysis was carried out. In this paper, the average annual precipitation in Florida of the United States is set as the benchmark 1, and 10 residual ratio curves with the change of relative precipitation from 0.3 to 3.0 are analyzed, as shown in Figure 18.

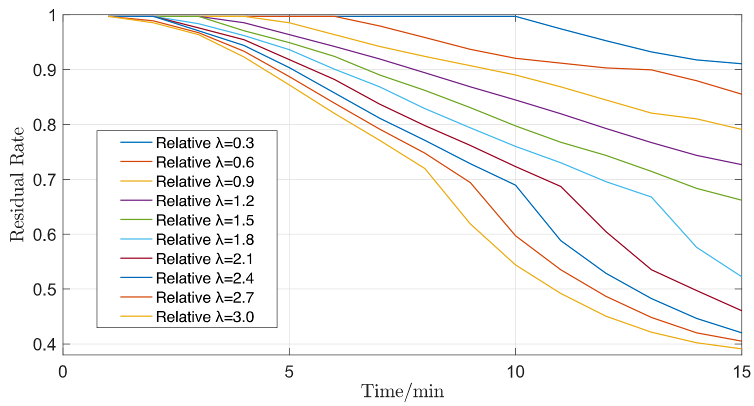


Figure 18: Sensitivity analysis about rainfall

Different from the previous similar curves, the span edge of the abscissa time is much smaller, because the residual ratio of sandcastle is already less than 0.4 when relative $\lambda = 3.0$ at 15min. It is meaningless to continue to study the 3-dimensional geometry of sandcastle as time increasing. The residual ratio of the corresponding curve at the same time $\lambda = 0.3$ was greater than 0.9, indicating that the sandcastle could maintain a good geometry in small rainfall. On the whole, the rate of sandcastle erosion is positively correlated with rainfall, indicating that the model is sensitive to rainfall, which is also conform with life experience.

7 Strengths and weaknesses

7.1 Strengths

- **Modeling Based on Stationary cellular automata.** The continuous model is unable to simulate the complex system such as sandcastle, but the cellular automata can simulate the spatiotemporal evolution of the complex system. After the stationary cellular automata runs for a certain time, the cellular space will tend to a fixed state that does not change with time, this represents an ideal 3-dimension prototype.

- **Results agree with common sense and experience.** This paper identifies the role of the rudiment. The facing water surface of sandcastle buffered the seawater

关注数学模型
获取更多资讯

and slowed down the seawater speed. Another surface is used as the main body of the sand castle to build a rich sand sculpture. The rudiment itself is consistent with the idea of building sand castle in real life.

- **Easy for implement.**This paper constructs a sand castle model based on cellular automata, thus transforms the complex solving process of 3d problem into the setting of the cell and the operation of the cellular automata. The self-organization characteristic of cellular automata means that the cellular itself will reach the optimal solution in an iterative way. This method guarantees the accuracy and efficiency of the model solution, which is easy to realize in programming.

7.2 Weaknesses

- **Inaccurate force analysis.**In models of scouring and floatage, force analysis of particular matter such as sand grains are oversimplified. Although the object of our study is the sand cell composed of sand grains, it is not completely accurate to use classical mechanical force analysis for establishing the model.

- **Lack of strong theory of optimal water-sand ratio.** Due to limited data and references about sand-water mixing, the formula of optimal water-sand ratio and its parameters are inferred from insufficient experience.

8 Conclusions

Based on the simulation result, here summaries the conclusions of our analysis. Destructive power from the environment including rising seawater and rainy weather is the vital threat to sandcastle foundation. Multiple models established on detailed mathematical analysis gives answers the required questions.

Primarily, the longest lasting 3-dimensional geometric sandcastle foundation shapes share similar appearance illustrated in Figure 16 and Figure 12. They are characterized with shell-shaped cross section and the three parts A,B and C. Generally speaking, sandcastle foundations with milder slopes specialize in resisting erosion of incoming seawater, while those with larger top has better rain-proof ability. It is surprising that after adequate times of iterations, all the initial sandcastle foundation transfer to shapes with similar structure which has been mentioned before.

Secondly, The best sand-to-water mixture proportion is around 2.63 or 2.70 in the rain, which is a result of trade-off between the increasing friction and liquidity when there are growing ratio of water in the rain.

Lastly, three measures can be taken to strengthen the sandcastle foundations,including the adjustment of building time, adding supporting structure and improving building materials. Those are all practical measures one can take in order to obtain a better sandcastle foundation.

关注数学模型
获取更多资讯

References

[1] Robert C. Beardsley ,Changsheng Chen ,and Qichun Xu2. Coastal flooding in Scituate (MA) : A FVCOM study of the 27 December 2010 noreaster[J].Joutnal of geophysical research: Oceans, V. 118, 60306045, 2013

[2] Bak, P., Tang, C. and Wiesenfeld, K.. Self-organized criticality: an explanation of 1/f noise. Physical Review Letters. 1987, 59 (4): 381384.

[3] Zhang YuHua, Yu Song, Tan GuangYu. Research on Thermal Force Coupling Behavior of 3D Milling Tools Based on Cellular Automata Method[J]. Aerospace precision manufacturing technology, 2018, 054(004):1-7,13.

[4] Ma Fei. The Movement and Nonlinear Analysis of Sediment Particle [D].TianJin University,2012.

[5] Lu KunQuan, Liu JiXing. Static and dynamic properties of granular matter(I) [J].PHYSICS,2004(10):713-721.

[6] LU KunQuan, Liu JiXing. Static and dynamic properties of granular matter(2II [J].PHYSICS,2004(10):713-721.

关注数学模型
获取更多资讯

Appendices

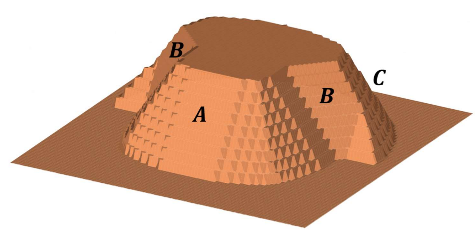
Magazine Article for Fun In The Sun

How to Make Your Sandcastle Last Longer

Have you ever been to seaside? Have you ever stacked sand sculptures? If so, you may wonder how to make a wet sand pile prototype, and then use imagination to create a dreamed castle. However, you have also encountered the sand castle washed away by the waves, wet and deformed by the rain. We have the empathy when you were looking at the destruction of your hard work. Our Team use use metathetical model to build the longest lasting sand castle prototype for you.

We ignored evaporation in wet sand and assumed that the sand castle has a fixed distance from the sea water. Using the physical principle analysis and mathematical tools, combined with the existing data, we simulated the most lasting sand castle prototype. This prototype is mainly composed of three parts: the front side of sandcastle foundation is blunt and its slope is gentle, which is stable to resist the erosion of the sea water; on both sides facing sea called the buffer part, whose slope is steep, playing a role in slowing down the speed of the sea water; on the back facing sea, it is the main part of the sand castle, its structure is similar with the front, but its volume can be very large, and the main structure of sand castle will be concentrated on this side.

The figure below is the model simulation diagram we have obtained. It should be noted that the volume of the impacted sandcastle front cannot be too small, otherwise it will not protect the back, and the width of the main body shall not exceed the buffer zone, otherwise it will be easily washed away by the waves.



Good models require good raw materials to achieve. Before building a sand castle, we need to mix dry sand and sea water into wet sand, because if the viscosity between sand and sand increase, the sand castle will naturally become more stable. Of course, if you have other materials, such as glue, you can also mix it into the sand, and the results will be better. If not, we have worked out the most durable volume ratio of water and sand for you, that is, 1litre of sand mixed with 0.38litre of water. According to this ratio, your sand castle can last for a long time even without the help of other tools.

Our simulations show that, with using our prototypes and materials, even if you're building a sandcastle for the first time, you can still make it pretty durable. What else can you worry about? Go to build the castle you imagine!

In addition to these, we have also analyzed the impact of wind, sunshine and other environmental factors on the most durable sand castle. Limited in length, this article will not elaborate it. If you want to know, you are welcome to subscribe to the next issue of *Fun in the sun*.

关注数学模型
获取更多资讯