Round or Browned?
Heat Distribution and Packing Efficiency
in Brownie Pans between a Circle and a Square

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Abstract

In this paper, we deal with packing efficiency and heat distribution problems arising from cooking brownies in an oven. We seek a pan shape that will maximize a combination of packing efficiency and even heat distribution. We consider three types of shapes which are "between" a circle and a square (regular polygon, squares with rounded corners, and "squircles") for latter analyzing. We model "browning" or "burning" at the edge of the pan using the heat equation; we approximate results with a random diffusion simulation. Then by applying algorithm from [3] and [4], we assess the packing efficiency of different shapes. We gather data for different types of shapes and fit a model to the data. Using this model, we determine the best pan shape to maximize packing efficiency and minimize browning when various weights are given to each criterion. If good heat distribution is weighted heavily, the circle is still the best shape out of all the ones we considered. In other cases, the hexagon is optimal according to our model. Our study may improve the design or choice of pans under different situations.

Contents

1   Summary

When cooking brownies in the rectangular pans, the corners tend to get over-heated. The circle has an even heat distribution, but it does not pack efficiently in an oven. We search for pan shape that *combine* even heat distribution and packing efficiency by considering shapes "in between" a square and a circle.

We develop models for heat distribution and packing efficiency in pans with different shapes. We apply our model to

1. regular polygons

2. squares with rounded corners

3. another square-circle hybrid we call a "squircle."

We collect data about each type of shape using each model and fit polynomial models to the data. Finally, we optimize a weighted average of heat distribution and packing efficiency. The structure of the paper is as follows.

- In Section 2, we explain our preliminary assumptions about the oven and the pans. We describe the three types of shapes we consider out of the very large search space of possible shapes.

- Section 3 discusses heat distribution. We use the heat equation to model the amount of "browning" or "burning" at the edges of the pan. We approximate the results of the heat equation in two dimensions using a random diffusion simulation. We use the simulate the amount of browning for various shapes.

- In Section 4, we model the packing efficiency of various shapes using an optimal nesting algorithm developed by [3] and [4]. Given a many-sided polygon, the algorithm circumscribes a hexagon which can tessellate the plane; it minimizes the area of the circumscribed hexagon. We apply the shape to regular polygons. We approximate the round shapes we are considering by many-sided polygons, and then apply the algorithm to them also.

- In Section 5, we combine the models in section 2 and 3 by introducing an objective function and assigning relative weights to heat distribution model and optimal nesting model. We find the best shapes out of the three categories and the best overall.

- Section 6 summarizes the results, assesses the strengths and weaknesses of our model, and makes suggestions for generalization and improvement.

2   Introduction

When cooking brownies, some people like square pans because they pack more efficiently in an oven, while others prefer circular pans because the brownies

are cooked more evenly on the edges. We want to find a "happy medium"–
something between a circle and a square that maximizes some combination of
packing efficiency and even heat distribution.

2.1   Preliminary Assumptions

We assume at the outset that *heat distribution is even throughout the oven.*
This is actually true for convection ovens which circulate hot air with a fan [1].
Conventional ovens distribute heat unevenly, but the unevenness is unrelated
to pan shape, so we do not consider it in our model. An immediate consequence
of this assumption is that the number and spacing of oven racks are irrelevant.
They will only multiply the total number of pans by an integer.

Regarding the pans, we assume:

1. *Pans cannot overlap in the oven.*

2. *All our pans are the same shape.* It is possible to consider combinations
   of shapes, but for simplicity, we assume the pans are the same.

3. *The pans are convex.* Concave pans have a significantly larger perimeter
   than convex pans, and hence they are more susceptible to browning at the
   edges. Many concave shapes do not pack well either.

2.2   Shapes to Consider

As it stands, the problem is still very broad because the search space of possible
shapes is so large. It is not only infinite, but even infinite-dimensional. A pan
shape can be given by parametrizing the boundary by *any* simple closed curve
$\mathbf{f} : [0, 1] \to \mathbb{R}^2$; thus, we are looking for an optimum over a *function space*.[1] The
search space is still infinite-dimensional, even if we assume the enclosed shape
has area 1 and its centroid is at the origin.

Rather than finding the best shape in general, we will consider three cate-
gories of shapes which are "between" a circle and a square in some sense. We
will optimize the solution for each category and compare the results.

2.2.1   Regular Polygons

First, we consider regular polygons (square, pentagon, hexagon, etc.). The for-
mula for a regular n-gon with area 1 centered at the origin, in polar coordinates,
is

$$r \le \frac{1}{\sqrt{n \tan \frac{\pi}{n}}} \sec\left(\theta \bmod \frac{2\pi}{n}\right).$$

We will optimize heat distribution and packing efficiency as a function of n.

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[1]We cannot apply standard variational techniques here either because the function we are
trying to minimize does not have a neat formula.

Figure 1: Squares with rounded corners for $k = 0.2, 0.4, 0.6, 0.8$.
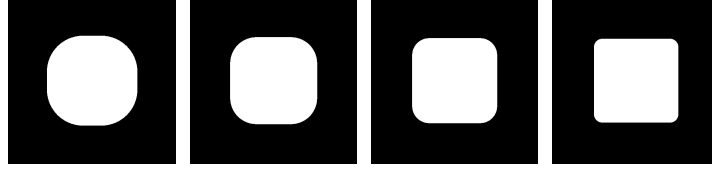
2.2.2   Squares with Rounded Corners

Second, we consider replacing the corners of a square with quarter-circles. We choose $k \in [0, 1]$ to represent the fraction of one side of the square the remains straight; m represents half the width of the original square. Thus, $k = 0$ gives a cirlce and $k = 1$ gives a square. The portion of the square with $x, y \geq km$ is replaced by a quarter circle centered at (km, km) and similar replacements are made at the other corners.. To normalize the area of the figure to one, we let $m = 1/\sqrt{4 - (4 - \pi)(1 - k)^2}$. Then the shape is given by

$$\begin{cases} |y| \leq m, & \text{if } |x| \leq km \\ |y| \leq km + \sqrt{(1-k)^2 m^2 - (|x| - km)^2}, & \text{if } km \leq |x| \leq m. \end{cases}$$

We will optimize our objectives as functions of k.

2.2.3   "Squircles"

Third, we consider a shape, which we will call a "squircle," which is a hybrid of a square and a circle. We obtained it by taking a weighted "average" of a square and circle in polar coordinates in the following way. A circle with area 1 is given by $r \leq 1/\sqrt{\pi}$. A square with area 1 is $r \leq \frac{1}{2} \sec \theta$ for $-\pi/4 \leq \theta \leq \pi/4$, with similar formulae in the other sectors of the plane. The area computation for a quarter of each shape is

$$\int_{-\pi/4}^{\pi/4} \int_0^{1/\sqrt{\pi}} r \, dr \, d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{\pi} \, d\theta = \frac{1}{4}$$

$$\int_{-\pi/4}^{\pi/4} \int_0^{(1/2)\sec\theta} r \, dr \, d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{4} \sec^2 \theta \, d\theta = \frac{1}{4}.$$

By taking a weighted average of the integrals on the right, we get another integral equal to $\frac{1}{4}$. That is, for $0 \leq k \leq 1$,

$$\frac{1}{4} = \int_{-\pi/4}^{\pi/4} \left(\frac{1-k}{\pi} + \frac{k}{4} \sec^2 \theta \right) d\theta = \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{(1-k)/\pi + (k/4)\sec^2\theta}} r \, dr \, d\theta.$$

Figure 2: Squircles for $k = 0.2, 0.4, 0.6, 0.8$.

Thus, $r \leq \sqrt{(1-k)/\pi + (k/4)\sec^2\theta}$ for $-\pi/4 \leq \theta \leq \pi/4$ (with similar formulae in the other sectors) gives a shape with area one, which is "between" a circle and a square; $k = 0$ gives a circle, $k = 1$ a square. In rectangular coordinates, the shape has the formula[2]

$$\frac{k}{4\max(x^2, y^2)} + \frac{1-k}{\pi(x^2 + y^2)} \geq 1.$$

We will optimize our objectives with respect to k.

3 Modeling Heat Distribution

3.1 Assumptions

We model the amount of "browning" or "burning" at the edge of a pan during baking. Our earlier assumptions guarantee that the heat distribution is the same for each of the pans.

We assume further that the food has a *uniform density* throughout baking, so that the shape of the browned area can be modeled by the *heat equation*. In bread, baking and crust formation is complicated because it involves not only heat flows but also moisture flows and evaporation [10]. The crust shape cannot be modeled by only the heat equation. The crust (dry region) and crumb (moist region) change size because of evaporation, as the crust is compacted by the expanding crumb. Thus, the crust does not vary in thickness near the corners.

These effects, however, are much less significant for brownies. Brownies lose much less water by weight than bread. Their density is more uniform, and so the burned area can be modeled more accurately by the heat equation. We assume at this point we are dealing with brownies or something with similar properties.

According to the heat equation, the heat $u(x, y, z, t)$ in a three-dimensional shape S satisfies

$$u_t = \alpha(u_{xx} + u_{yy} + u_{zz}) \text{ for all } (x, y, z) \in S.$$

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[2]The formula is for $(x, y) \neq (0, 0)$. The point $(0, 0)$ is always inside the shape.

When the object is surrounded by liquid or gas at a different temperature, the boundary conditions are given by the thermal transfer equation:

$$u_t = k(T_{\text{ext}} - u) \text{ for all } (x, y, z) \in \partial S$$

where T_{ext} is the external temperature and k is a heat transfer coefficient. This equation is still messy computationally, and it is difficult to find tools for solving it over a nonrectangular region.[3] We make further simplifying assumptions for the sake of computation.

Specifically, we treat the equation in two dimensions rather than three. Since our model only deals with the two-dimensional profile of the brownie pan and we do not know its height, we cannot model in three dimensions accurately. Rather, we assume the pan has a uniform height and that horizontal and vertical effects of heat are roughly independent, that is, $u(x, y, z, t) \approx f(x, y, t) + g(z, t)$. We concentrate on the effects of the horizontal function f. Thus, we simulate the heat equation in two dimensions, with heat only entering from the edges.

3.2 Simulating Diffusion

Rather than solving the 2D heat equation directly, we model its behavior with a much simpler algorithm. The key insight to our model is that the heat equation governs *more* than just heat. It also describes diffusion of particles by random walking or more generally by *Brownian motion*–for example, the diffusion of gases, the spread of particles through a liquid, or the flow of liquid through a porous solid [9] [6].

For each type of pan under consideration, we generate a picture of a white shape on a black background. We allow the black pixels to diffuse throughout the shape like gas filling an empty space or a water filling a sponge. At each time step,

- We let each black particle choose a random horizontal or vertical direction.

- If the adjacent pixel in that direction is white, we move the black pixel; otherwise, we leave it where it is.

- After we have iterated over the whole picture, we refill the original black area (i.e. the part that was black at $t = 0$) with black pixels.

This last action simulates gas or liquid coming from the outside to take the place of the particles that moved into the original white area. At the end of the simulation, the "burned" part of the shape is the portion of the original white area that has become black.

Because the movement of particles operates by the same physical laws and equations as the movement of heat, we can also view the algorithm as simulating

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[3]Even Wolfram Mathematica can only solve PDEs on a rectangle, and it cannot solve the heat equation with this type of boundary condition, even on a rectangle.

Figure 3: Application of the diffusion algorithm to a pentagon at 300×300 pixels, at $t = 50, 100, 200, 300$.
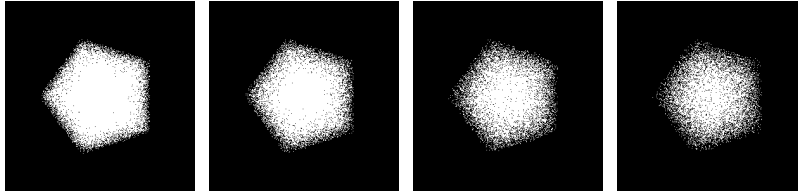
heat. The heat is simply the concentration of black particles at each point. The white has a heat of 0 and the black a heat of 1.[4]

Up to this point, we have made many approximations, not all of which are justified theoretically, but the results of the algorithm are quite reasonable (see Figure 3). Notice, however, that although we chose to work in two dimensions, our diffusion model would work just as well in three dimensions.

3.3 Results for Polygons, Rounded Squares, and Squircles

For each n-gon we ran a set up a simulation with the following parameters:

- The resolution was 500×500 pixels.

- The area of the shape was 1, where the frame represented $-1 \leq x, y \leq 1$.

- We performed 25 time steps.[5]

- We recorded the fraction of the original white area that had become black, taking the median over 19 simulations.

We plotted the results as a function of n and fit a model to the data. Since regular polygons approach a circle as $n \to \infty$, we assume the browning of each polygon approaches the browning of a circle asymptotically. We assume they decrease toward that value as the shapes become less and less susceptible to heat from the perimeter. We wanted our interpolating function to have the same properties. Thus, instead of a polynomial model, we used a linear combination of n^{-1} and n^{-2} added to the value obtained for a circle. We fit the polynomial for $n = 4$ to 19 using Wolfram Mathematica's least-squares algorithm. The error was less than 0.067 or 0.87% of the expected value.

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[4]Because the heat equation does not change if we scale or translate u and T_{ext} by the same constants, the specific temperature values do not affect the structure of the solution. The simulation also assumes $k = 1$ (which can be obtained analytically by scaling the time units) and some other constant value for α.

[5]It is actually 50 iterations over the picture. Because we iterated over the image row by row, there were situations where a black pixel might move and then be considered for movement again. To prevent this flaw from skewing the results, we iterated forward and then backward at each time step.

Figure 4: Browning amounts for polygons; n is the number of side and b is the percent browned.
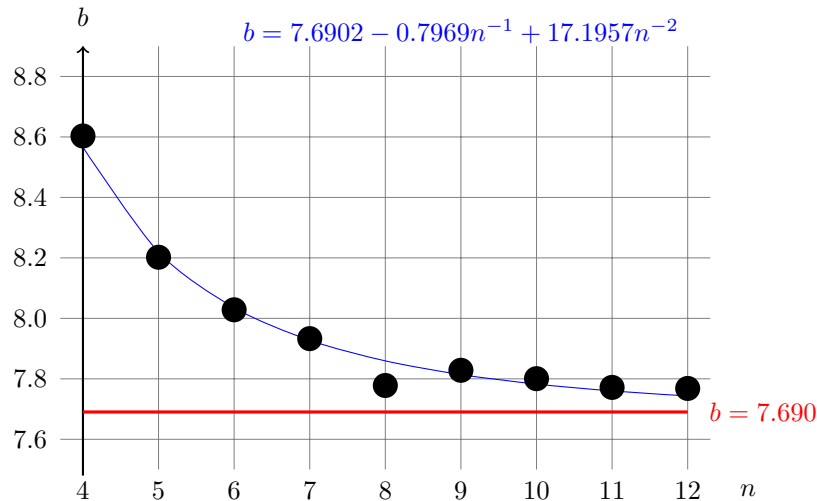
We applied the same procedure to squares with rounded corners, this time writing the results as functions of the parameter k. We sampled k at multiples of 0.1. We fit a quadratic model to the data; at each point the error was less than 0.5% of the value. We applied the same procedure to squircles; k was sampled at multiples of 0.1. The quadratic approximation was within 0.2% of the expected value. As we would expect, **in each of the three categories, the circle optimized heat distribution.**

4 Packing Pans

We measure the packing efficiency of polygons by how well they tessellate the plane. Of course, *an arrangement which tessellates the plane well may not fill a finite oven well.* If the oven is large enough, it *will* be a good approximation to the actual efficiency. If the oven is small, the tessellation will often be inefficient at the edges because the tessellation will not have smooth edges. However, in practice, most cooks do not place pans too near the edge of the oven because the oven walls may affect the heat distribution. But even if someone wants to fill the oven all the way to the edges, our results will still provide an upper bound for packing efficiency in a finite oven.

Figure 5: Browning amounts for squares with rounded corners; b is the percent browned. Recall $k = 0$ is a circle and $k = 1$ is a square.
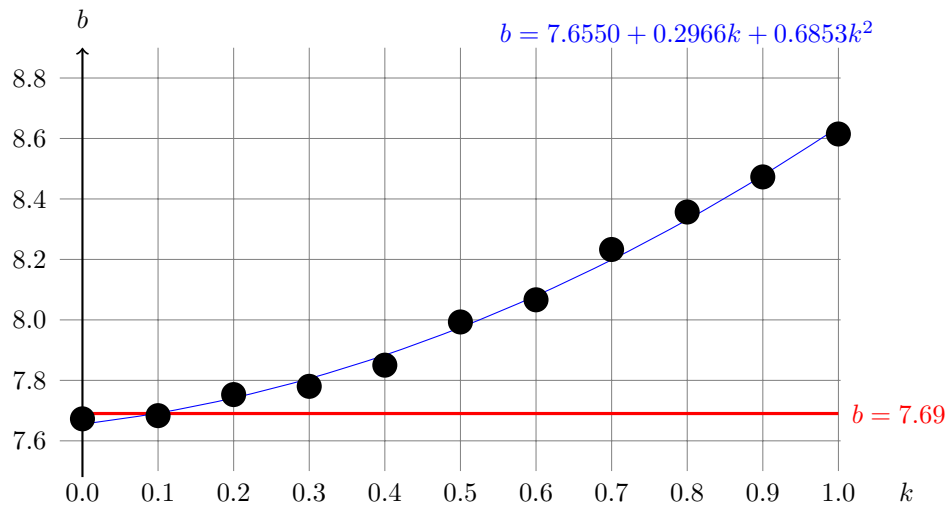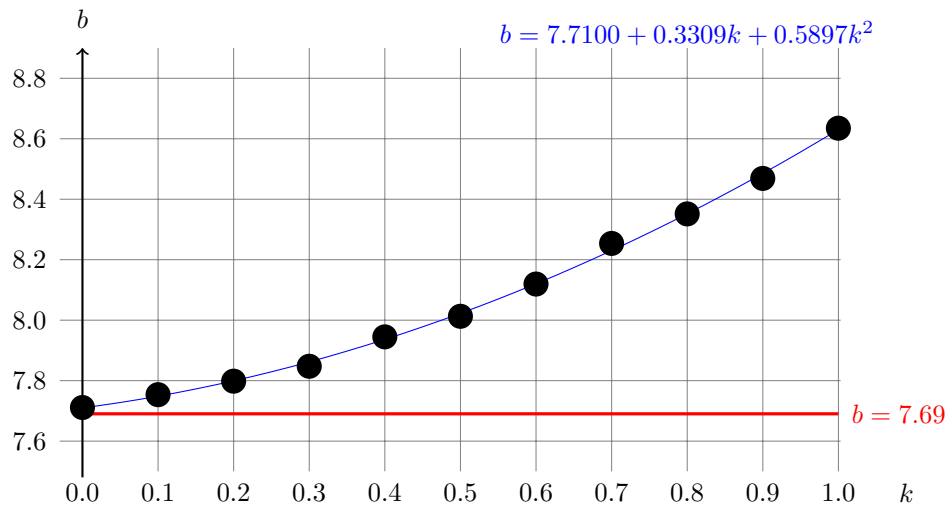
$$b = 7.6550 + 0.2966k + 0.6853k^2$$

$$b = 7.69$$

Figure 6: Browning amounts for squircles; b is the percent browned.

$$b = 7.7100 + 0.3309k + 0.5897k^2$$

$$b = 7.69$$

4.1   Optimal Nesting Algorithm

To measure the packing efficiency, we implement the optimal nesting algorithm described by Dori and Ben-Bassat in their papers [3] and [4], which produces a near-tessellation of the plane out of a given convex polygon with six or more sides. The algorithm circumscribes a tessellating or "paving" hexagon around the original figure with near-minimal added area. We use *hexagons* because they have enough sides to preserve area efficiently, and yet they can easily be adjusted to tessellate the plane.

The resulting hexagon P_6' will tessellate the plane and provides a good approximation to the optimal packing of the original polygon. After applying the algorithm, we measure the packing efficiency of the original shape by how much area the algorithm added. The packing efficiency the area ratio of the original polygon to the circumscribed hexagon, or

$$\text{efficiency} = \frac{A(P_n)}{A(P_6')}.$$

We summarize the algorithm briefly here. For a detailed explanation and a proof of the correctness and efficiency of the results, see [3] and [4]. In explaining the algoithm we use the following notation:

- P_n: an n-sided convex polygon.

- $\bar{P}_6$: a hexagon where opposite sides are *parallel*.

- P_6': a hexagon whose opposite sides are *parallel* and have *equal length*.

- S_i: The sides of P_n is denoted by $S_1, S_2, ..., S_n$. S_i is the i-th side of P_n.

- V_i: The vertices of P_n is denoted by $V_1, V_2, ..., V_n$ in order as we proceed around the edge of the polygon.

- A_i: The interior angles of P_n is denoted by $A_1, A_2, ..., A_n$, according to the number of the vertices. A_i is the corresponding angle of S_i

- R_i: The minimum adding area to replace S_i, S_{i+1} in P_n by a new side S_i' crossing V_i. S_j' is always out of P_n.

- $BO(P_n)$: the basic order of a P_n. P_n is of *ofbasicorderk*–denoted by $BO(P_n) = k$–if it has exactly k sides, $S_{j_1}, S_{j_2}, ..., S_{j_k}$, such that

$$A_{j_1} + A_{j_i+1} < \pi \text{ for } j_i = 1, 2, , ..., k$$

By the notation above, the basic structure of the algorithm is as follows:

1. Given an n-sided polygon P_n, circumscribe a hexagon P_6 with the smallest area.

    - Define the unit operation as circumscribing P_m by P_{m-1} with minimum of R_j for all $j = 1, 2, ..., m$.

Figure 7: Application of the minimum fit algorithm to an 11-gon. The first picture shows the hexagon P_6, the second the tessellating hexagon P_6'.
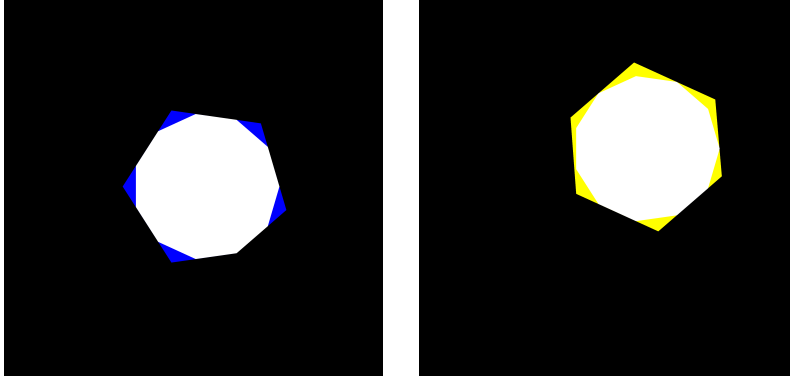

- To obtain P_6 from P_n, we proceed iteratively by reducing one side each time. Begin with the unit operation. Every time when we do a unit operation to P_m to get P_{m-1}, if $BO(P_{m-1}) \leq BO(P_m)$ (actually it's never the case that $BO(P_{m-1}) \leq BO(P_m)$ shown by [4]), continue to do basic operation until we get P_6. Otherwise go to the next step.

- When $BO(P_{m-1}) > BO(P_m)$, get P_{m-2} by finding the minimum of the minimum adding area when replacing all sides in P_n not in P_{m-1} to a new side S_k'. Then go back to the previous step.

2. Given the hexagon P_6, circumscribe a new hexagon P_6' such that opposite sides are *parallel* and have *equal length*. We find P_6' in two steps:

  - Circumscribe P_6 with $\bar{P}_6$.
  - Circumscribe $\bar{P}_6$ with P_6'.
  - The P_6' is what we want.

With this algorithm, we can easily deal with any convex polygon to produce the minimum circumscribed P_6' to pave the plane.[6] However, the algorithm above will not directly applicable to the rounded squares, squircles, or any rounded shape. We circumscribed rounded shapes with a many-sided polygon P_n and then applied the polygon packing algorithm to get an approximated result.

We circumscribe a convex shape with curve in it to P_n by first partitioning the curve into n pieces with equal length and then choosing the tangent line of

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[6]For P_4 and P_6, the packing efficiency is perfect. For P_5, we determined an efficient arrangement by hand.

Figure 8: Packing efficiency for regular polygons; e stands for efficiency (%). The red line represents the packing efficiency of a circle.
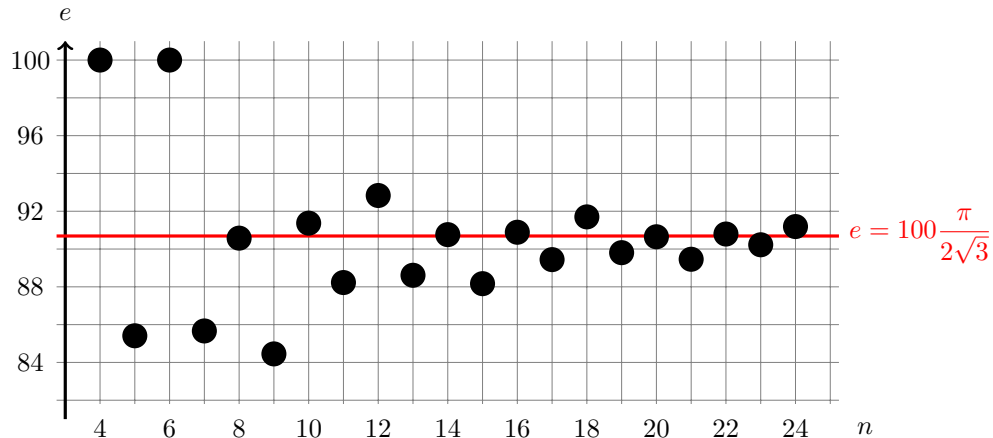
the middle point in the piece-curve to produce the circumscribed P_n. This is a naive approach which may not ensure the produced polygon P_n has minimum additional area (out of all possible n-gons that circumscribe the original figure). However if we make n large enough, the error is negligible. We used P_{128} to model rounded squares P_{100} for squircles.

4.2   Results for Polygons, Rounded Squares, and Squircles

The packing efficiencies of regular polygons are shown in Figure 8. The values are erratic, especially for low values of n. Indeed, the square and hexagon can tessellate the plane perfectly, but the pentagon in betwen them cannot. Even in an infinite plan, packing is *fundamentally discrete*. There is no hope for a smooth approximating curve as there was for heat distribution. However, if n is large enough, the n-gon is well-approximated by a circle, and so its packing efficiency will be close to that of the circle.

For squares with rounded corners, the data are well approximated by a quadratic (error less than 1.8, which is 1.9% of the expected value), although not as well as some of the other data sets. We interpolated the squircle data with a quadratic as well. The maximum error was 0.615, or 0.65%.

5   Optimizing Heat and Packing Simultaneously

We now combine our models to find the optimal shape in each category. Our objective function is a weighted average of packing efficiency (with weight p) and even heat distribution (with weight $1 - p$).

Before calculating the function, we normalize our data by shifting and scaling the dependent variable so that the output for a circle becomes 0 and the output

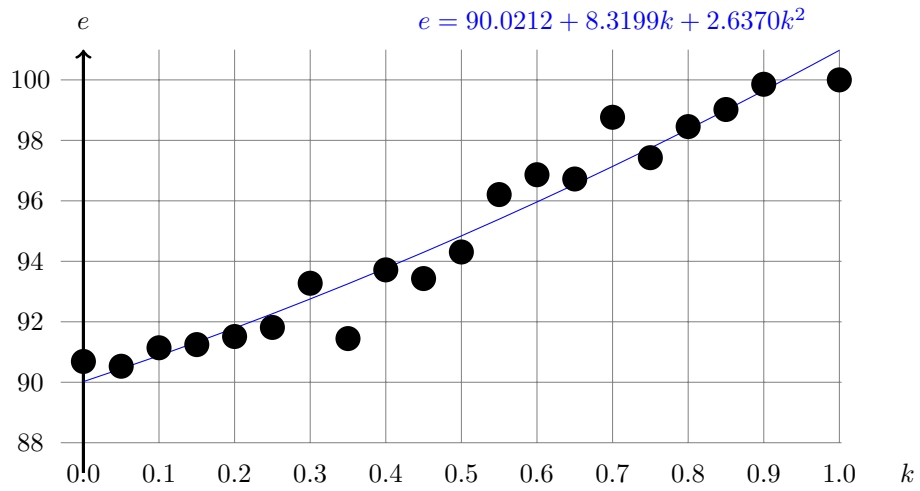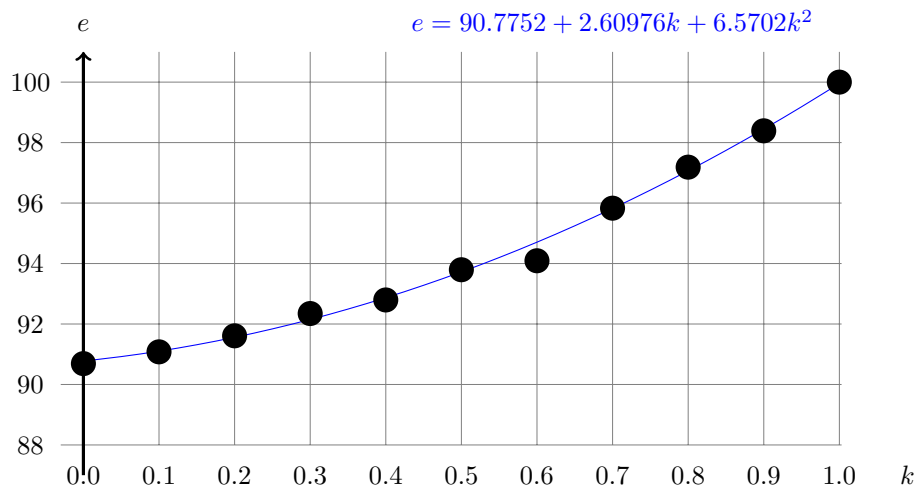Figure 9: Packing efficiency for rounded squares with interpolating quadratic; e stands for efficiency (%).

$$e = 90.0212 + 8.3199k + 2.6370k^2$$

Figure 10: Packing efficiency for squircles with interpolating quadratic; e stands for efficiency (%).

$$e = 90.7752 + 2.60976k + 6.5702k^2$$

for a square 1. Thus, the browning data range from 0 (circle) to 1 (square), and so do the other data. Because we want to maximize one function and minimize the other, we subtract the browning equation from one. Thus, our objective is to maximize

$$\text{objective} = p(\text{packing}) + (1 - p)(1 - \text{browning}).$$

Because the amount of browning and packing efficiency was difficult to predict for the polygons, we considered each polygon separately and wrote a formula for the objective function in terms of p in each case. For each value of p between 0 and 1 at steps of 0.001, we found the polygon which maximized the objective function. We considered polygons with $n = 4$ to 19. The results were as follows:

$$n_{\max} = \begin{cases} 16, & \text{if } 0.000 \le p \le 0.192 \\ 18, & \text{if } 0.193 \le p \le 0.286 \\ 6, & \text{if } 0.287 \le p \le 1.000. \end{cases}$$

In general, the hexagon is by far the best polygon according to our model. This is because only the hexagon and the square have optimum packing efficiency, and the hexagon has a smaller amount of browning than a square.

For the rounded squares, we wrote the objective function in terms of the quadratic polynomials we had fit to the data:

$$\text{objective}(k) = (1 - p)(1.0385 - 0.3249k - 0.7508k^2)$$
$$+ p(-0.0718 + 0.8936k + 0.2832k^2).$$

On the interval $[0, 1]$, the objective attained its maximum at 0 (the circle) for $p \le 0.2666$. For $p \ge 0.5557$, the best shape was the square. Between 0.2666 and 0.5557, the optimal value of k was given by

$$k_{\max} = \frac{-1.2186p + 0.3249}{2.0681p - 1.5016},$$

which was obtained by setting the derivative of the objective function to zero. The results are shown in Figure 11.

When we normalized the interpolating polynomials for the squircle data, the two equations were almost identical:

$$\text{browning}(k) = 0.02112 + 0.36246k + 0.646037k^2$$
$$\text{packing}(k) = 0.00916 + 0.2803k + 0.70571k^2.$$

The function $p(\text{packing}(k)) + (1 - p)(1 - \text{browning}(k))$ always achieves its maximum over $[0, 1]$ at one of the endpoints. That is, either the circle or the square is the best option. If a higher priority is placed on non-browning, the circle is best, and otherwise, the square is best. The squircle design is not effective for this optimization.

Figure 11: Optimal values of k for rounded squares as a function of p.
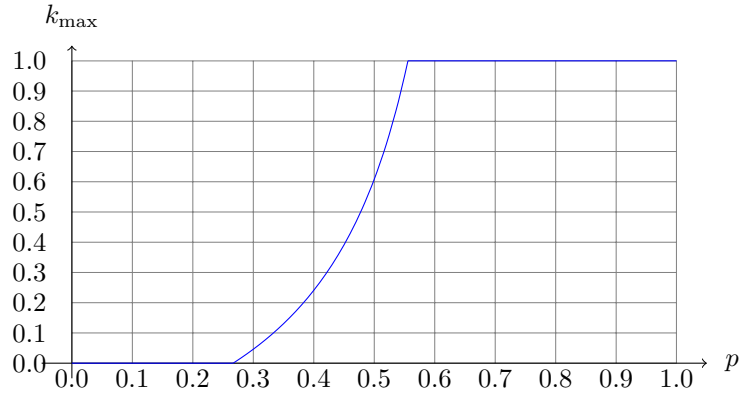
We now combine our data to find the optimal shape out of all the ones we considered, for various values of p. The results are shown in 12.[7] The optimal shape according to our model depended on the value of p. For small p, the circle was best. For a brief interval from 2.666 to 0.2702, the rounded square was optimum. After that, the hexagon took the cake.

$$\text{best shape} = \begin{cases} \text{circle,} & 0 \leq p \leq 0.2666 \\ \text{rounded square,} & 0.2666 \leq p \leq 0.2702 \\ \text{hexagon,} & 0.2702 \leq p \leq 1. \end{cases}$$

6   Conclusions

6.1   Summary

We divided the brownie pans problem into two subproblems:

- Modeling the heat distribution by a diffusion simulation: We simulated the amount of burned material at the edge of the pan.

- Packing the shapes efficiently in the plane: We estimated how efficiently shapes could pack by circumscribing them with a minimum-area tessellating hexagon.

Then, we optimized the even heat distribution and efficient packing simultaneously. Our objective function was a weighted average of packing efficiency and good heat distribution.

We applied our model to three types of shapes:

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[7]The blue curve for rounded squares should approach the red line for the square; in fact, at $p = 0.5574$, it is off by 0.481 because of experimental error (the packing data for the rounded squares differed the most from the expected results).

Figure 12: Values of the objective function for various optimal shapes.
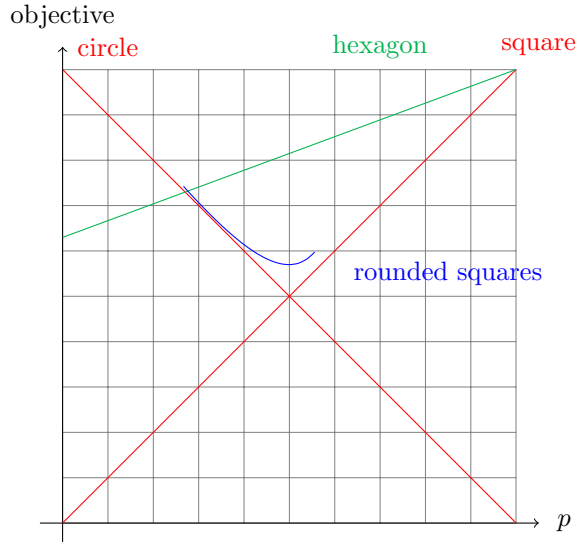
1. Polygons with $n \geq 4$ sides.

2. Squares with rounded corners. The amount of rounding was specified by a parameter k, where $k = 0$ gives a circle and $k = 1$ a square.

3. "Squircles"–a circle-square hybrid. The amount of rounding was specified by a parameter k, where $k = 0$ gives a circle and $k = 1$ a square.

6.2   Results

We gathered data for packing efficiency and good heat distribution. We fit quadratic models to the data for rounded squares and squircles, but the data for polygons was too erratic. We normalized the data by scaling and translating the dependent variable so that the results fit in a range from 0 to 1. For packing, 0 represented the circle (worst) and one the square (best). For heat distribution, 0 represented the square (worst) and 1 the circle (best). We maximized a linear combination of packing efficiency (with weight p) and good heat distribution (with weight $1 - p$) for each type of shape.

The results were as follows:

- Among the polygons, the hexagon was best for $p \geq 0.287$. For smaller p, many-sided polygons were most efficient.

- Among the rounded squares, the circle was best for $p \leq 2.666$, the square for $p \geq 0.5557$. In between, the best rounded square was given by $k_{\max} = (-1.2186p + 0.3249)(2.0681p - 1.5016)$.

- Among the squircles, the maximum was always either the circle or the square.

- *Among all the shapes considered, the circle was best for small p and the hexagon for large p.*

For low values of p, the circle was the best shape, but in general, the hexagon took the cake.

6.3   Assessment of the Model

In the following subsections, we will discuss some general strengths and weaknesses of our model, generalizability to other scenarios, and possible improvements to the model.

6.3.1   Strengths

1. Our model accurately simulates the heat equation with very simple computations.

2. It provides a good measure of how well a polygon fills space.

3. The packing algorithm is efficient by itself, taking linear time.

6.3.2   Weaknesses

1. The nesting algorithm computes packing efficiency based on the assumption of infinite plane. However, we use compute the efficiency over the finite plane, which will contain some degree of error.

2. Our diffusion model does not take into account the third dimension–the height of the pan.

3. The packing model does not handle non-convex shapes.

4. The model does not consider internal heat and moisture in the food could effect the heat diffusion.

5. Our model would benefit from more accurate data.

6.3.3   Generalizability

Although we only considered three types of shapes, our approach can be applied more generally.

1. The diffusion simulation works for any shape.

2. The simulation could easily be extended to three dimensions.

3. The packing algorithm works for any convex polygon with 6 or more sides.

4. We can apply the packing algorithm to other convex shapes after approximating them with polygons.

6.3.4   Improvements

1. Considering other types of shapes.

2. Simulating the heat equation in three dimensions, testing various values for the height of the pan.

3. More precise algorithms for solving the heat equation over an irregular region.

4. Simulation of both heat and moisture in the baked goods.

5. Calculating packing efficiency over a finite region rather than the infinite plane.

# 7   Advertisement for a Magazine

Pan Pan Pain

Imagine you are baking brownie for lots of people in a big party. It would be rude to ask guests to wait so long for your gourmet. So you might want to bake as much as you can. In front of you, there are different kinds of pans, saying circular, rectangular, or even hexagonal. You need to make some decisions. You may choose the rectangular pans to pack things very tightly in the oven to cook many at the same time, but you also know you might desperately find later that your food totally ruined by the burnt part at the corner. Or you may choose the circular pans as you usually cooked for yourself before, but in front of you, there might be a longer and longer line of people, waiting your food and complaining about how slow you did that. A little hard to choose, eh?

According to our recent research on packing efficiency and heat distribution of the pan in baking, you might find a reasonable way to get an optimized solution to your previous dilemma.

Pans usually come in circular or rectangular shapes, or rectangles with rounded corners. You could imagine all kinds of other shapes as well–for instance, why not use a pentagon or hexagon? Which pan is the best?

The packing problem, actually, is a space efficiency question; on the other hand, the quality of baked food is related to how evenly the heat is distributed in the pan. In our research, we modeled these two aspects separately and considered a weighted combination of them. These are our results.

Regarding packing efficiency, if we consider a large oven with small pans (so that the boundary will not influence the arrangement too much), the packing efficiency is related to the shape of the pans, as the picture shows. If we only consider space efficiency, its nice to choose rectangular and hexagonal pans. Also when the shape is closer and closer to a circle, we find that the packing efficiency approaches to a certain number.
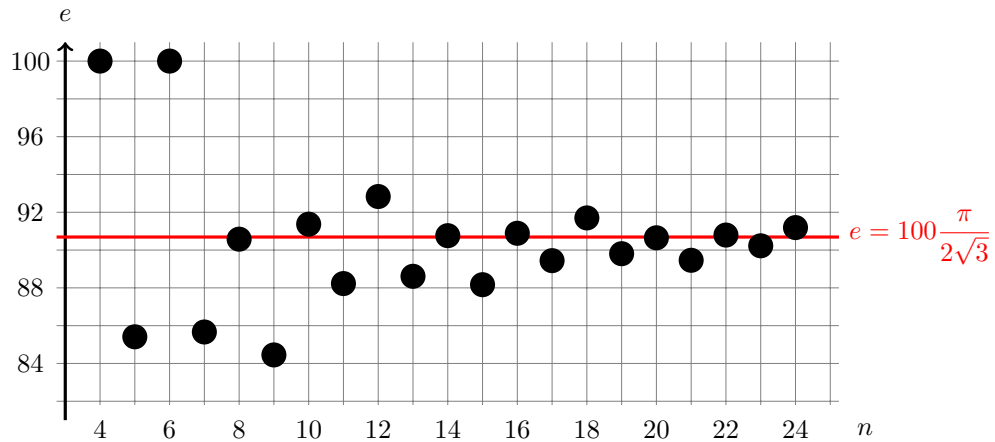
How about the square with round corners? We find a much easier rule. The packing efficiency always increases as the pan becomes more square.

In terms of heat distribution, a circle is always the best shape. It has least amount of edge to be burned.

If we weighted the two criteria, what will happen? Too get a easy conclusion, lets consider just three cases, which is you want to have highest space efficiency, most even heat distribution, or both of them (half and half).

From our research result, for highest space efficiency, you might choose the rectangular or hexagonal pans. If you want to cook food more delicious, circular pans must be your best choice. If we consider both, hexagons are the best polygonal pans in general. If you value even heat distribution very highly and if you HAPPENED to have those shapes on hand, 16-gon shape pan and 18-gon shape pan would be better. For a square pan with rounded corners, if we value good heat distribution and packing efficiency equally, the best shape is when the straight part of the side is 0.6 times the width of the pan.

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