Piping Hot Weather

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A_p	Cross sectional area of the pipe
A_{sn}	Cross sectional area of the sprinkler nozzle
C	Drag coefficient on a moving sphere
γ	Drag force on the water droplet
ρ	Density of water
r_{drop}	Radius of water droplet
ξ	Flowrate in the pipe
m	Mass of a water droplet
n	Number of nozzles
g	Acceleration due to gravity
v_p	Velocity of water in the pipe
v	Exit velocity of the water out a nozzle
$S_x(t)$	Horizontal distance of a projected droplet at time t
$S_y(t)$	Vertical distance of a projected droplet at time t
s	Maximum horizontal distance from a nozzle reached by a droplet
$\sigma(r)$	Height of water per unit time at a radial distance from the nozzle
K	Parameter of σ
λ	Parameter of σ
$\tilde{\Sigma}(r)$	Sum of contributions of watering from two nozzles
$\Sigma(r)$	$\tilde{\Sigma}(r)$ expressed in centimeters per hour
r_e	Effective radius of water jet
H	Cut-off parameter
d	Distance between sprinklers
d_p	Distance between sprinklers on pipe
fr	Fraction of hour that water is switched on
sh	Number of shifts to fill a rectangle
T	Time that pipe is in each position
T_{min}	Cumulative time to satisfy minimum water conditions

Contents

1 Introduction

Irrigation is a fundamental concern for farmers in regions with a low annual rainfall. So important is it that the California Irrigation Institute ([1]) lists as one of its objectives the 'increased appreciation by the California public of the importance of irrigation to California's economy'. The Department of Water Resources in California ([2]) urges people to be constantly planning for drought.

Numerous irrigation systems are available. An analysis of some of these is given by [3]. One involves positioning sprinkler heads on a long aluminium pipe, which is moved periodically so as to cover the entire field with sufficient water. This method is relatively cheap and straightforward to apply and so is suitable for smaller farms. However this can be labour intensive as the farmer has to move the equipment around the field. Our aim in this report was to find an optimum configuration for the number of sprinklers on the pipe so that the effort required by the farmer could be minimized.

2 Problem Analysis

An 80 by 30 metre field is to be irrigated using a 20 metre pipe. The number of sprinklers and pipe movements are to be decided so as to minimize the work needed to be carried out by the farmer. This optimisation problem is subject to the constraints that every part of the field requires at least 2 centimetres of water every 4 days, while no part should receive more that 0.75 centimetres per hour.

The first task was to model the sprinkler and the resulting distribution of water. This distribution should be as uniform as possible to avoid watering too much or too little, and also because it is desirable to have an even irrigation over the field. A rotating spray nozzle will send out a circular 'umbrella' of water when considered over a period of rotation. Since the optimisation then essentially involves packing circles into a rectangle, overlap is unavoidable and this had to be taken into account and incorporated into the model for the watering.

Once this model was completed, we would be left with a situation looking roughly as shown below: The number of sprinkler nozzles could be then

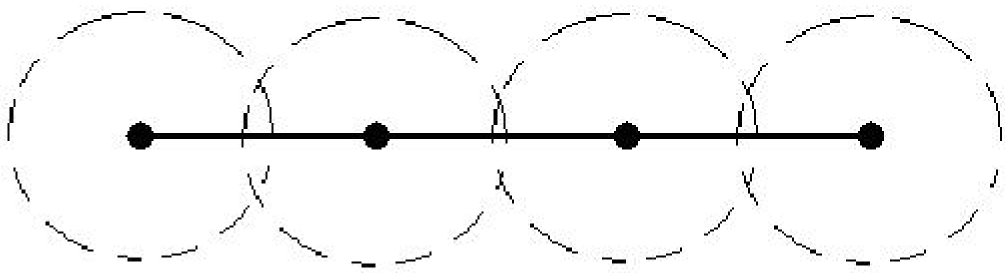

Figure 1: Example with 4 sprinklers on the pipe, each emitting a circular spread of water

varied subject to conditions from our model, and the resulting distribution of water could be determined. With this information, an algorithm for an efficient watering can be devised, in which the farmer can be given step-by-step instructions on where to place the pipe and how long to leave the water flowing.

3 Assumptions

- We do not consider the time taken to move and set up the pipe and water source. Instead we define an optimum efficient algorithm to be the one which minimizes the number of moves needed to water every part of the field with the required amount.

- We assume the sprinkler nozzle sends out a jet of water which can be modelled as a projectile. The nozzle head rotates so that in one period of rotation, a circular area is watered.

- We then ignore the actual design of the sprinkler head and model the system as a pipe with the sprinklers acting as smaller branches off the main pipe, as shown below:

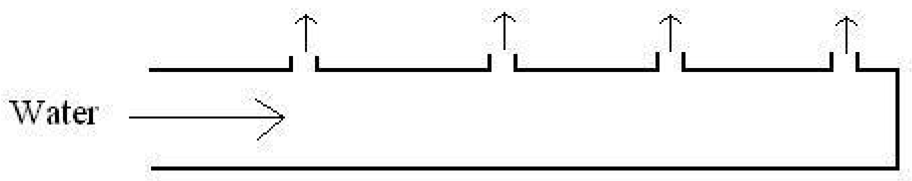

Figure 2: Design of the pipe and sprinklers

- As can be seen from the diagram, we aren't concerned with how the pipe is connected to the water source and this also means that we are free to move the 20m pipe around the field without physical restrictions. We take the flow rate of 150 liters per minute to be the rate entering the pipe. We keep the other end closed and it is then reasonable to approximate a constant pressure through the pipe.

- When modelling the water flow and the water leaving the sprinkler, we ignore viscosity and turbulence and assume that no wind is blowing.

- So as to design our algorithm for getting the correct amount of water to every part of the field, we presume that no rain falls.

- We assume that the pipe is equipped with a timer control, which can stop and start the flow of water at preprogrammed times.

4 Model

4.1 Fluid Through Pipe

As mentioned earlier, we are supposing that the pressure remains constant. We use a standard continuity equation, which can be found in [4] for example:

$$A_p v_p = A_{sn} v \tag{1}$$

where A_p and A_{sn} are the cross-sectional areas of the pipe and the spray nozzles respectively and v_p and v are, respectively, the velocity of the flow through the pipe and at the nozzle. By assuming that the flowrate ξ would be equally distributed among the n nozzles, equation (1) becomes

$$A_{sn} v = \frac{\xi}{n} \Rightarrow v = \frac{\xi}{n A_{sn}} \tag{2}$$

This gave us v, the exit velocity of the water out of any nozzle.

4.2 A Projectile Problem

We next model the emerging jet of water as a projectile of water droplets. By focusing on a single droplet with radius r_{drop} and mass m, we obtained the following differential equations for the distance travelled by the droplet along the x and y axes respectively, where the nozzle is at the origin:

$$mS_x''(t) + \gamma S_x'(t) = 0 \tag{3}$$

$$mS_y''(t) + \gamma S_y'(t) + g = 0$$

The γ parameter represents drag on the droplet, given in terms of the density rho, drag coefficient C and the surface area of the droplet A, by

$$\gamma = \frac{1}{2}\rho C A^2 \tag{4}$$

More details on this equation can be found in [5]. We were interested in finding s, the maximum horizontal distance travelled from the nozzle by a droplet. For this maximum, the launch angle is $\frac{\pi}{4}$. Therefore the initial conditions for the equations are as follows, using the exit velocity v from equation (2):

$$\begin{aligned} S_x(0) = 0 \quad \text{and} \quad S'_x(0) = \frac{v}{\sqrt{2}} \\ S_y(0) = 0 \quad \text{and} \quad S'_y(0) = \frac{v}{\sqrt{2}} \end{aligned} \tag{5}$$

4.3 Distribution of Water

4.3.1 Height per unit time

The water from the sprinkler is distributed by a rotating nozzle. Ignoring the rotation of the nozzle momentarily, the volume of water per unit time, through the nozzle, is spread with a certain function of height $\sigma(r, w)$ at a position along the radius r and along the width w (between $-\Delta w$ and Δw) of the spray.

As rotation is taken into account, and the system is considered in time spans of whole periods, the width w becomes the circle with radius r, i.e. $w \to 2\pi r$, a function of the radius r. This means that the height per unit time at a particular point is a function of the variable r only: $\sigma(r)$.

4.3.2 Function

For the function $\sigma(r)$, we have decided to take an arbitrary function of r and adjust the shape to match an expected output of water.

We feel intuitively that $\sigma(r)$ would have a rounded peak after which it would decay exponentially, to avoid too much loss of water due to evaporation

, mist effects, etc. This suggested the equation:

$$\sigma(r) = Kre^{-\lambda r}$$

To avoid a zero value at $r = 0$, we then shifted the function to the left, by an amount a fifth of the position of the inflection point giving:

$$\sigma(r) = K(r + \frac{2}{5\lambda})e^{-\lambda(r+\frac{2}{5\lambda})}$$

The output of water now looks as follows:

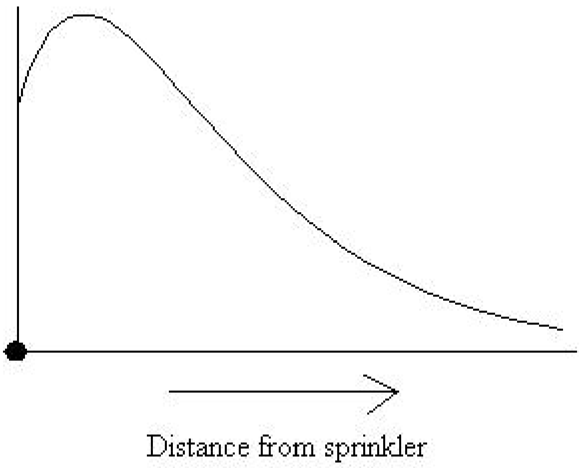

Figure 3: Graph of $\sigma(r)$

4.3.3 Scaling along r

To scale our new function $\sigma(r)$ along r, we use the result of the 'Projectile Problem' from earlier. Since the maximum range of a water droplet has quite a high arc to follow, we decided that the sprinkler would now be designed to spray much of the water this far and chose to set the distance roughly twice the distance that $\sigma(r)$ seems to drop off at. Keeping the figures easy to work with, we decided on a value giving an exponential constant of $\lambda = 8/s$, giving:

$$\sigma(r) = K(r + \frac{s}{20})e^{-\frac{8}{s}(r+\frac{s}{20})} \tag{6}$$

4.3.4 Scaling of height

To find the multiplicative constant K, giving the vertical scaling, we need the total volume of water per unit time within the circle of radius s. This is found by integrating $\sigma(r)d\theta dr$, over the whole circle and between $r = 0$ and $r = s$, i.e.:

$$\int_0^s \int_0^{2\pi} \sigma(r) d\theta dr$$

This should then be equal to the flow rate exiting the sprinkler $\frac{\xi}{n}$:

$$\frac{\xi}{n} = \int_0^s \int_0^{2\pi} K(r + \frac{s}{20})e^{-\frac{8}{s}(r+\frac{s}{20})} d\theta dr \tag{7}$$

$$= 2\pi K \int_0^s (r + \frac{s}{20})e^{-\frac{8}{s}(r+\frac{s}{20})} dr \tag{8}$$

4.3.5 Cut-off point

In reality, there is a distance from the sprinkler, beyond which the majority of the water is unlikely to reach the ground at all, a fact which was mentioned earlier. As a result we have decided to define a cut-off point, beyond which the height of water per unit time actually hitting the ground may be considered negligible. This cut-off point gives an 'effective radius' r_e, defined by a normalized $\sigma(r)$:

$$\frac{\sigma(r_e)}{r_e} = H \tag{9}$$

where H is an arbitrary number chosen to give a reasonable solution.

4.3.6 Sum of contributions (overlap)

Since overlapping will be necessary to ensure that every part of the field receives the minimum required amount of water, we need a method of finding the contributions of two consecutive sprinklers to the ground between them.

To simplify matters, we have assumed that a sum of the two contributions along the line joining the two sprinklers would be a close enough approximation to the rest of the overlapping area.

Since we are assuming the sprinklers to be equivalent in they're output of water, the functions $\sigma(r)$ are the same for each, however, setting one sprinkler at $r = 0$, the effect of the other has to be reversed and translated by the distance between the two sprinklers d. This gives the sum:

$$\tilde{\Sigma}(r) = \sigma(r) + \sigma(d - r)$$

Multiplying this by a conversion factor, to get our answer in terms of cm per hour, we get:

$$\Sigma(r) = (3.6 \text{ x } 10^5)(\sigma(r) + \sigma(d - r)) \tag{10}$$

4.4 Condition for Minimum Sprinkler Separation

Intuitively, if the system is to be efficient, for every pair consecutive sprinklers there should not be a significant amount of water being spread from one sprinkler beyond the other. Since we have already defined an 'effective radius' r_e, beyond which the water is unlikely to reach the ground, this is a natural condition to take for the minimum distance between two consecutive sprinklers.

4.5 Restriction for Number of Sprinklers on Pipe

Since the pipe given is 20 metres long and there are n sprinklers spread across the pipe, by the Pigeon-Hole Principle, at least one of the distances between two of the sprinklers will be greater than or equal $\frac{20}{n-1}$, which we call d_p.

From the previous section, we have a condition for the minimum distance between two sprinklers. Therefore, if this minimum distance r_e is greater than the distance from the pipe, then this configuration cannot be allowed.

This gives us a condition for whether or not a system with n sprinklers on the pipe is effective or not:

$$r_e \leq \frac{20}{n-1} \tag{11}$$

4.6 Filling The Field

In finding the best way to position the pipe at every move, for a chosen value of n, the problem was reduced initially to filling the field with circles of radius r_e, under the conditions:

1. Every point must be contained in a circle,
2. No circle should contain the centre of another,
3. Circles must be grouped linearly, with n circles in each group,
4. Overlapping is allowed at the field edges.

The first and third conditions are obvious, the second comes from the condition for the minimum distance between two sprinklers and the last is a requirement for the edges of the field to get enough water.

4.6.1 Pipe

To begin with, the n sprinklers were assumed to be spread evenly along the pipe, with sprinklers at each end. If there is not overlapping between the sprinklers, then the pipe will need to be moved along its length, a fraction of the distance between the sprinklers to fill the gap. To find how many times this must be done, we need to divide d_p by r_e and take the ceiling, which we're calling the shift of n:

$$sh(n) = \ulcorner \frac{d_p}{r_e} \urcorner$$

(For values of n where there is overlapping already, the shift is equal to 1).The fraction the pipe will have to be moved is $\frac{d_p}{sh}$. For values of n where there is overlapping already, the shift is equal to 1.

4.6.2 Rectangle

Next, we want to find the rectangle around the pipe within which every point is covered by a circle (i.e. being watered). To find the dimensions of this rectangle, we need to find two parameters: x which represents the half the distance between two sprinklers and y the perpendicular distance from the pipe to the edge of the rectangle. We can obtain y easily from Pythagoras theorem, giving:

$$x = \frac{1}{2}\frac{d_p}{sh} = \frac{10}{sh(n-1)}$$

$$y = \sqrt{{r_e}^2 - x^2} = \sqrt{{r_e}^2 - (\frac{10}{sh(n-1)})^2}$$

It can be seen easily from the diagram that for a shift of 1, the length of the rectangle is x metres extra at each end of the pipe and so is $(20 + 2x)$ metres. It should also be clear that the height of the rectangle is $2y$ metres. The dimensions of the rectangle are then for a shift of 1 are then:

$$(20 + 2x)x(2y)$$

If the shift is greater than 1, the length of the rectangle is simply $(20 + \frac{20}{n-1})$, giving a rectangle of dimensions:

$$(20 + \frac{20}{n-1})x(2y)$$

4.6.3 Filling a field

To find the minimum amount of moves needed to water the whole field, the most efficient way of covering every part of the field with rectangles of these dimensions must be found. Due to the fact that the effort required by the rancher is to be minimized, we have ignored configurations where the pipe is not aligned parallel to one of the sides and also tried to reduce the number of times that the direction of the pipe was changed. Also, by the fourth condition above, the rectangles are allowed to run over the edges of the field. In practice this was just a matter of trial and error, under these conditions.

4.6.4 Adjusting for overlapping

To spread the "overlapping" over the whole field, the rectangles are reduced in size and. Since the length of these new rectangles involves the fixed distance between the sprinklers on the pipe, this has to be fixed for all of the rectangles in the field. The heights of these new rectangles however, simply represents the distance between different positions of the pipe, and so may have a number of different values for different rectangles.

These rectangles then define where on the field the pipe has to be positioned as well as the distance between the sprinklers on the pipe.

4.7 Timing

4.7.1 Distances between sprinklers

To find the amount of time that the sprinklers need to be left in each position, we need to know the different distances between the consecutive sprinkler positions. These distances can be calculated from the adjusted rectangles from the previous section.

Firstly, the distance between the sprinklers on the pipe needs to be recorded.

For two rectangles lying side by side, the distance is half the sum of the rectangle heights.

For rectangles lying perpendicular to each other, the distance is half the sum of the distance between the sprinklers on the pipe and the height of the rectangle whose side is touching the top of the other.

4.7.2 Avoiding saturation

The sum $\Sigma(r)$ must be plotted with respect to r for each of the distances d and the overall minimum Σ_{min} and maximum Σ_{max} values found.

If the maximum value is greater than 0.75 cm per hour, then the water can only be left on for a fraction of an hour, each hour. This fraction fr is found by dividing 0.75 by the maximum value:

$$fr = \frac{0.75}{\Sigma_{max}}$$

To find the amount of time the pipe must stay in each location, the minimum watering of 2cm in 4 days must be satisfied. This time T_{min} is found by dividing 2mm by Σ_{min}:

$$T_{min} = \frac{2}{\Sigma_{min}}$$

However, if $fr < 1$ then the time that the sprinklers are left in each position must be:

$$T = T_{min}.fr$$

The total time for irrigating the whole field can then be found by multiplying the number of moves by the time the sprinklers are left in each position T. This, obviously, must be less than 96 hours (4 days) for the irrigation to be successful.

5 Results and Algorithm

5.1 Numerical Results

5.1.1 The Projectile

From our projectile model of a single droplet of water being emitted from a spray nozzle with a 0.6 cm inner diameter we obtained the following solutions to the differential equations given by (3):

Along the x direction, the trajectory of the water droplet was

$$S_x(t) = C_1 + C_2 e^{-\frac{\gamma}{m}t}$$

Along the y direction, the trajectory of the water droplet was

$$S_y(t) = C_3 + C_4 e^{-\frac{\gamma}{m}t} - \frac{mg}{\gamma}t$$

From the initial conditions (5), the constants C_i were found to be

$$C_1 = \frac{mv}{\gamma\sqrt{2}}$$

$$C_2 = -C_1$$

$$C_3 = \frac{m}{\gamma}\left(\frac{v}{\sqrt{2}} + \frac{mg}{\gamma}\right)$$

$$C_4 = -C_3$$

The velocity v is calculated from equation (2). The mass of a water droplet is $m = \frac{4}{3}\pi\rho r_{drop}^3$. The constant γ was given by equation (4). The value for the drag coefficient of a sphere was taken from [6] to be $C = 0.47$. Finally, g is the acceleration due to gravity.

Using these equations we solved $S_y(t) = 0$ to determine the time t at which the water droplet would reach the ground. It was assumed that the spray nozzle was at ground level. Taking the flow rate $\xi = 0.0025\ m^3\ s^{-1}$,

$A_{sn} = 2.85 \times 10^{-5}\ m^2$, $g = 9.81\ m\ s^{-2}$, $\rho = 1000\ kg\ m^{-3}$, $r_{drop} = 0.0003\ m$, it was found that the droplet reached the ground after $t = 2.54682\ s$. Using this value for t the range of the water droplet was estimated by evaluating $S_x(t)$ at $t = 2.54682\ s$. The range of the water droplet was then found to be $s = 31.8109\ m$.

5.2 Valid Values of n

To find which values of n are valid, a programme was constructed. The programme also calculated the values of r_e for the values of n which are valid. The results from this programme are shown below:

n		r_e
2	invalid	
3	valid	0.563074
4	valid	1.57976
5	valid	3.26629
6	invalid	
7	invalid	
8	invalid	
9	invalid	
10	invalid	

5.3 $n = 3, 4, 5$

The shift sh and the dimensions of the initial rectangles were then calculated for the three values of n:

n	sh	rectangle dimensions
3	18	20.356 x 0.480 metres
4	5	21.333 x 2.864 metres
5	1	25.000 x 4.202 metres

Since the $n = 4$ and $n = 3$ cases take 5 and 18 moves respectively, to fill a rectangles which are smaller then that of the rectangle for 1 move of the $n = 5$ case, the total number of moves are going to be much greater then that of the $n = 5$ case. As a result, these cases were dismissed at this stage.

5.3.1 Field positions for $n = 5$

The most effective way of filling the field with the initial rectangles for $n = 5$ was found, by trial and error, to be the configuration shown below.

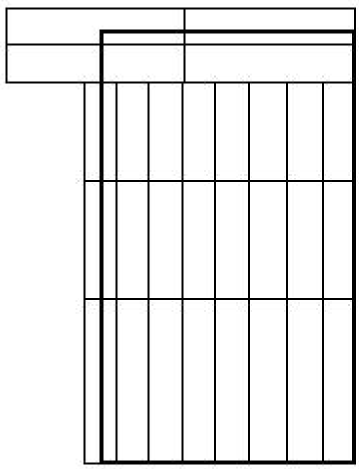

Figure 4: Optimum configuration

The sizes of the rectangles was then adjusted to fit into the field, giving the following positions of the pipes:

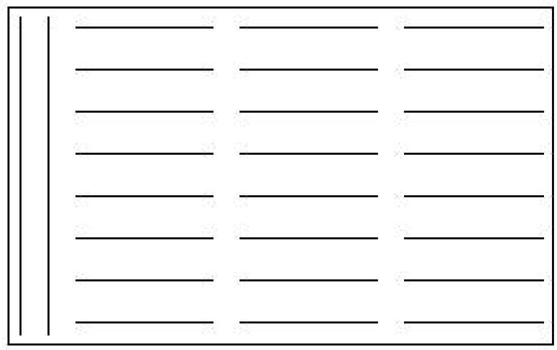

Figure 5: Position of the pipes for the watering scheme

5.3.2 Timing for $n = 5$

There were three different values of d found:

$$d_1 = 3.75m$$

$$d_2 = 4.00m$$

$$d_3 = 4.83m$$

A plot was then made of $\Sigma(r)$ for these three values of d:

The maximum and minimum values were found for each d and the overall maximum and minimum values determined:

$$\Sigma_{max} = 0.84 \text{ per hour}$$

$$\Sigma_{min} = 0.67 \text{ per hour}$$

From these, it was found that the water could only be switched on for 53 minutes each hour and that the pipe had to remain in each position for 3 hours and 20 minutes.

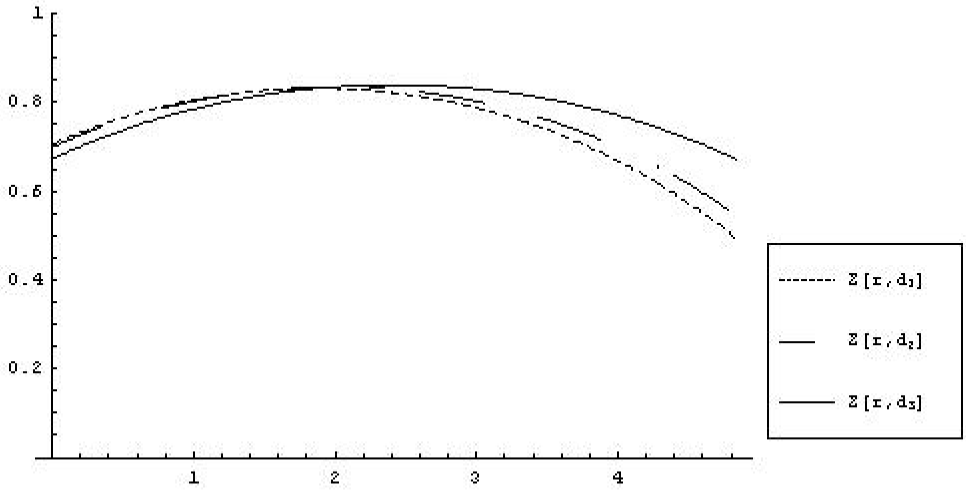

Figure 6: Values of Σ for the different d_i

5.4 Algorithm

The figures included here show how the pipes should be laid out in the field for maximum efficiency with respect to water coverage, number of moves and time for watering. For this configuration the 20 m section of pipe must be moved only 28 times every 4 days. The figure below shows the 28 positions, in which there are two overlaps on the left side of the arrangement.

The two graphs at the end of this section show the distances between the

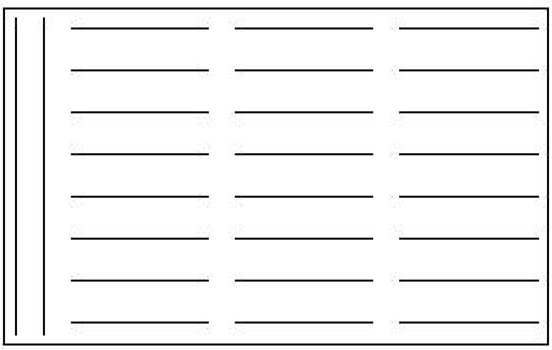

Figure 7: Position of the pipes for the watering scheme

pipe positions and the order and direction of the moves.

The water must be switched on only for 53 minutes of every hour which gives the most efficient use of water and delivers the maximum amount of water to the crops. In what follows we assume that the time taken to move the pipe is negligible compared to the time the water is switched on.

To utilize this excellent system the farmer must proceed as follows:

- Set up the pipe with the hose at the first point, as indicated on the diagram.
- Set up a timer such that water is being emitted from the pipe for only 53 minutes of every hour.
- Leave the pipe in place for three hours and 20 minutes.
- After three hours and 20 minutes move the pipe to the next position as indicated on the diagram.
- Repeat until all 28 positions have been visited and the field has been watered satisfactorily.

The idea is that the farmer, given the spacings between the pipes, would be able to follow the arrows on the diagram, and apply the watering schedule to obtain maximum watering efficiency.

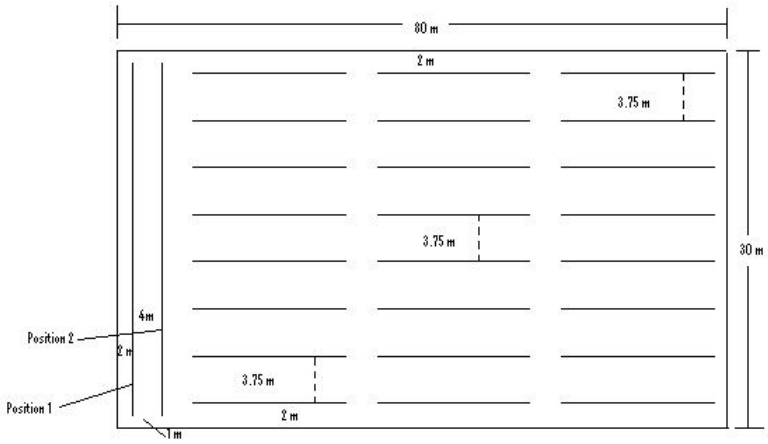

Figure 8: Distances between the pipe positions

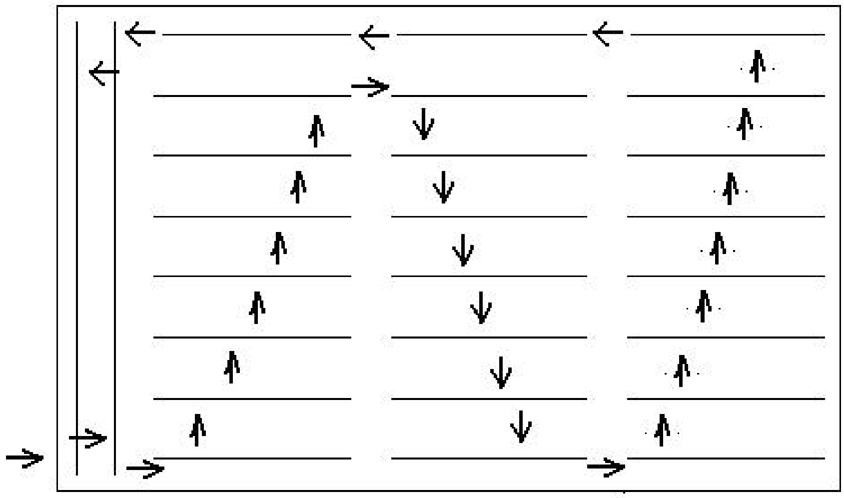

Figure 9: Order for the positions of the pipes

6 Strengths, Weaknesses and Possible Improvements

- In taking part in this competition one must be realistic of what can be achieved in the allowed time. However, one must also realize that a model of a highly complex system created in four days will not represent the physical situation in its entirety. To this unfortunate end we declare that our model is not the most general. However, it is possible that it can be generalized with further work.

- One of the key points of our model is that it yields a very straightforward algorithm. We give unambiguous instructions to the rancher on how to efficiently irrigate the field. The process is simple and easy to carry out, since it only involves moving a pipe a fixed distance in one direction.

- We also take into account the overlap of the water from different sprinklers and thus ensures that the water is evenly distributed and that no part of the field is over-watered or under-watered.

- The calculations we have performed are simple and easily implemented in any number of computer languages. We have included, for your perusal, the code we implemented on the Mathematica computer package as an appendix.

- Given the simplistic nature of our model concessions had to be made to describe the behaviour of the flow at the sprinklers and inside the pipe. In reality it cannot be assumed that the pressure would stay at a fixed constant value all the way along the pipe for an extended period of time. This assumption then gives us an unrealistic view of the velocity of the water as it leaves the sprinklers.

- One improvement that could be made to this model is an analysis of how the pressure would vary along the pipe. In particular we could analyse how the pressure would be affected by the number of sprinklers along the pipe. How would the pressure vary at the sprinkler-pipe interface? These are very complicated problems and in reality their analysis would require a detailed numerical solution of the Navier-Stokes equations.

- Another improvement that could be made is an analysis of the velocity of the water after it leaves the sprinkler. Given that the pressure and the flow rate are both very large this would mean that we are in the domain of turbulent flows. A highly complex and highly specialised area of research.

7 Conclusion

At the outset we were asked to invent a scheme to irrigate a field 80 metres by 30 metres subject to the constraints that no part of the field would get more than 0.75 cm of water per hour and also that every part of the field should get at least 2 cm of water every four days. We were also asked to determine the number of sprinklers that would be needed to fulfil these restraints as well as the spacing between the sprinklers. Also it would be advantageous if the scheme did not require too much labour for the rancher.

In the course of our analysis we have found that for 5 sprinklers spaced 4.83 metres apart we satisfy the requirements of the problem. We have shown that by turning the water on for 53 minutes of every hour in 3 hour 20 minute sessions, as explained by our algorithm, each section of the field gets no more than 0.75 cm of water per hour and that every part of the field gets at least 2 cm of water every 4 days.

Given that the rancher only has to move the pipe once every 3 hours and 20 minutes, and he need only do it 28 times, then it can be assumed that the process we have designed is not labour intensive and therefore satisfies all the requirements set forth by the problem at the outset.

References

[1] http://www.caii.org/

[2] http://watersupplyconditions.water.ca.gov/background.cfm

[3] http://www.ext.nodak.edu/extpubs/ageng/irrigate/ae91w.htm

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[5] Fulford G., Forrester P., Jones A. (1997),*Modelling with Differential and Difference Equations*,University Press, Cambridge

[6] http://www.aerodyn.org/Drag/tables.html#stream