第二章 极限

<海涅定理> $\lim\limits_{x\to x_0} f(x)=A \Leftrightarrow$ 对任何趋近于x_0且取不到x_0的数列$\{x_n\}$，$\lim\limits_{n\to\infty} f(x_n)=A$.

<无穷大量> $\underset{(a>1)}{\log_a n} < \underset{(a>0)}{n^a} < \underset{(a>1)}{a^n} < n! < n^n$

<等价无穷小> ($x\to 0$时) $x\sim\sin x\sim\tan x\sim e^x-1\sim\ln(1+x)$ $\cos x\sim 1-\frac{x^2}{2}$ $(1+x)^a\sim 1+ax$

只能在因式中使用.

练习2.1

1. (1) $\because \left|\frac{3n^2+2n+1}{2n^2-3}-\frac{3}{2}\right| = \left|\frac{4n+11}{4n^2-6}\right| = \left|\frac{4n+11}{2(2n^2-3)}\right| < \frac{|5n|}{2n^2}$ (当$n>11$时)

令 $\frac{5n}{2n^2}=\frac{5}{2n}<\varepsilon$. $\therefore n>\frac{5}{2\varepsilon}$. 取$N=\max\{[\frac{5}{2\varepsilon}]+1, 11\}$.

则对$\forall \varepsilon>0$. $\exists N=\max\{[\frac{5}{2\varepsilon}]+1, 11\}$. 只要$n>N$ 则 $\left|\frac{3n^2+2n+1}{2n^2-3}-\frac{3}{2}\right|<\varepsilon$.

$\therefore \lim\limits_{n\to\infty}\frac{3n^2+2n+1}{2n^2-3}=\frac{3}{2}$.

(2). $2^{-\sqrt{n}}\in[\frac{1}{2},1)$ $\therefore |2^{-\sqrt{n}}-1|=1-2^{-\sqrt{n}}\in(0,\frac{1}{2}]$.

对$\forall 0<\varepsilon<1$. 令$1-2^{-\sqrt{n}}<\varepsilon$. $\therefore n>\left(\frac{\ln 2}{\ln\frac{1}{1-\varepsilon}}\right)^2$

取 $N=1+\left[\left(\frac{\ln 2}{\ln\frac{1}{1-\varepsilon}}\right)^2\right]$ 只要$n>N$, 则$|2^{-\sqrt{n}}-1|<\varepsilon$. $\therefore \lim\limits_{n\to\infty}2^{-\sqrt{n}}=1$.

(3) 对$\forall\ 0<\varepsilon<\frac{\pi}{2}$. $\exists N=[\tan(\frac{\pi}{2}-\varepsilon)]+1$ 只要$n>N$.

即 $n>\tan(\frac{\pi}{2}-\varepsilon)\Rightarrow \arctan n>\frac{\pi}{2}-\varepsilon \Rightarrow |\arctan n-\frac{\pi}{2}|<\varepsilon$ $\therefore \lim\limits_{n\to\infty}\arctan n=\frac{\pi}{2}$.

(4) 由习题1(A)-2 ($p21$): $n!\geq n^{\frac{n}{2}}$. $\therefore \frac{1}{\sqrt[n]{n!}}\leq\frac{1}{\sqrt{n}}$. 对$\forall\varepsilon>0$,

$\therefore$取$N=[\frac{1}{\varepsilon^2}]+1$. 只要$n>N$. 即 $n>\frac{1}{\varepsilon^2}\Rightarrow\frac{1}{\sqrt{n}}<\varepsilon\Rightarrow\frac{1}{\sqrt[n]{n!}}\leq\frac{1}{\sqrt{n}}<\varepsilon$.

$\therefore \lim\limits_{n\to\infty}\frac{1}{\sqrt[n]{n!}}=0$

(5). $n=[(\sqrt[n]{n}-1)+1]^n=1+n(\sqrt[n]{n}-1)+\frac{n(n-1)}{2}(\sqrt[n]{n}-1)^2+\sum\limits_{k=3}^{n}C_n^k\cdot(\sqrt[n]{n}-1)^k$.

$\therefore n>\frac{n(n-1)}{2}(\sqrt[n]{n}-1)^2$ $\therefore (\sqrt[n]{n}-1)^2<\frac{2}{n-1}$ $\Rightarrow \sqrt[n]{n}-1<\sqrt{\frac{2}{n-1}}$.

令$\sqrt{\frac{2}{n-1}}<\varepsilon$. 有$n>1+\frac{2}{\varepsilon^2}$ 取$N=[\frac{2}{\varepsilon^2}]+2$.

$\therefore$对$\forall\varepsilon>0$. $\exists N=[\frac{2}{\varepsilon^2}+2]$. 只要$n>N$. 就有$\varepsilon>\sqrt{\frac{2}{n-1}}>\sqrt[n]{n}-1=|\sqrt[n]{n}-1|$. $\therefore\lim\limits_{n\to\infty}\sqrt[n]{n}=1$.

南開大學 作业纸

系别______ 班级______ 姓名______ 第 页

(6) $0<(1+\frac{1}{n^2})^n-1=\sum_{k=0}^{n}C_n^k\cdot\frac{1}{n^{2k}}-1=\sum_{k=1}^{n}C_n^k\frac{1}{n^{2k}}=\sum_{k=1}^{n}\frac{n(n-1)\cdots(n-k+1)}{k!\cdot n^{2k}}<\sum_{k=1}^{n}\frac{\overbrace{n\cdot n\cdots n}^{k个}}{k!\cdot n^{2k}}$

$=\sum_{k=1}^{n}\frac{1}{k!\cdot n^k}<\sum_{k=1}^{n}\frac{1}{n\cdot k!}=\frac{1}{n}+\frac{1}{n}\sum_{k=2}^{n}\frac{1}{k!}<\frac{1}{n}+\frac{1}{n}\sum_{k=2}^{n}\frac{1}{k(k-1)}=\frac{1}{n}+\frac{1}{n}\sum_{k=2}^{n}(\frac{1}{k-1}-\frac{1}{k})$

$=\frac{1}{n}+\frac{1}{n}(1-\frac{1}{n})=\frac{2}{n}-\frac{1}{n^2}<\frac{2}{n}$.　　令$\frac{2}{n}<\varepsilon$：$n>\frac{2}{\varepsilon}$.

对$\forall\varepsilon>0$. 取$N=[\frac{2}{\varepsilon}]+1$. 只要$n>N$. 就有$\frac{2}{n}<\varepsilon$. $\therefore\left|(1+\frac{1}{n^2})^n-1\right|<\varepsilon$.

$\therefore\lim_{n\to\infty}(1+\frac{1}{n^2})^n=1$.

(7). $\because 0<\alpha<1$. $\therefore$对$\forall x>1$, $x^\alpha<x$.

$\therefore (n+1)^\alpha-n^\alpha=n^\alpha\left[(1+\frac{1}{n})^\alpha-1\right]<n^\alpha(1+\frac{1}{n}-1)=n^{\alpha-1}$

令$n^{\alpha-1}<\varepsilon$. 即$\frac{1}{n^{1-\alpha}}<\varepsilon$. 即$n>\frac{1}{\sqrt[1-\alpha]{\varepsilon}}$.

对$\forall\varepsilon>0$. $\exists N=[\frac{1}{\sqrt[1-\alpha]{\varepsilon}}]+1$. 只要$n>N$. 就有$n^{\alpha-1}<\varepsilon$ 即$|(n+1)^\alpha-n^\alpha|<\varepsilon$.

$\therefore\lim_{n\to\infty}((n+1)^\alpha-n^\alpha)=0$.

2. (1) 对$\forall A>0$. 取$N=2([A]+1)$ $\therefore N^{(-1)^N}>A$. 即$\{n^{(-1)^n}\}$无界. 故其发散.

(2). 设$\lim_{n\to\infty}\sin n=A$. ($A$为固定常数).

则$\lim_{n\to\infty}(\sin(n+2)-\sin(n))=0$　$\therefore\lim_{n\to\infty}2\cos(n+1)\cdot\sin 1=0$　$\therefore\lim_{n\to\infty}\cos(n+1)=0$ $\therefore\lim_{n\to\infty}\cos n=0$

又$\cos(n+1)=\cos n\cos 1-\sin n\sin 1$　$\therefore\lim_{n\to\infty}\sin n=0$

$\therefore\lim_{n\to\infty}\cos n=\lim_{n\to\infty}\sin n=0$. 与$\cos^2 n+\sin^2 n=1$矛盾. $\therefore\{\sin n\}$发散.

练习2.2

1. $\because \lim\limits_{n\to\infty} x_n = a$ $\therefore$ 对 $\forall \varepsilon>0$. $\exists N$ 当 $n>N$ 时. $|x_n-a|<\varepsilon$.

$\therefore \big||x_n|-|a|\big| < |x_n-a| < \varepsilon$. $\therefore -\varepsilon < |x_n|-|a| < \varepsilon$.

$\therefore |x_n| > |a|-\varepsilon$. 取 $\varepsilon = \frac{|a|}{2}$ $\therefore |x_n| > \frac{|a|}{2}$.

2. $\{x_n\}$ 收敛、$\{y_n\}$ 发散 $\Rightarrow$ ① $\{x_n+y_n\}$ 发散 ② $\{x_n y_n\}$ 收散不定.

3. (1) 原式 $= \lim\limits_{n\to\infty} \dfrac{\frac{1}{3}\cdot(-\frac{2}{3})^n+\frac{1}{3}}{(-\frac{2}{3})^{n+1}+1} = \dfrac{\frac{1}{3}\lim\limits_{n\to\infty}(-\frac{2}{3})^n+\frac{1}{3}}{\lim\limits_{n\to\infty}(-\frac{2}{3})^{n+1}+1} = \dfrac{1}{3}$

(2) 原式 $= \lim\limits_{n\to\infty} \dfrac{\frac{1}{\sqrt{n+1}+\sqrt{n}}}{\frac{2}{\sqrt{n+2}+\sqrt{n}}} = \lim\limits_{n\to\infty} \dfrac{\sqrt{n+2}+\sqrt{n}}{2(\sqrt{n+1}+\sqrt{n})} = \lim\limits_{n\to\infty} \dfrac{\sqrt{1+\frac{2}{n}}+1}{2(\sqrt{1+\frac{1}{n}}+1)} = \dfrac{1}{2}$

(3) 原式 $= \lim\limits_{n\to\infty} \left[(\sqrt{n+2}-\sqrt{n+1})-(\sqrt{n+1}-\sqrt{n})\right]\sqrt{n^3} = \lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n+2}+\sqrt{n+1}} - \dfrac{1}{\sqrt{n+1}+\sqrt{n}}\right)\sqrt{n^3}$

$= \lim\limits_{n\to\infty} \dfrac{\sqrt{n}-\sqrt{n+2}}{(\sqrt{n+2}+\sqrt{n+1})(\sqrt{n+1}+\sqrt{n})}\cdot\sqrt{n^3} = \lim\limits_{n\to\infty} \dfrac{-2\sqrt{n^3}}{(\sqrt{n+2}+\sqrt{n+1})(\sqrt{n+1}+\sqrt{n})(\sqrt{n+2}+\sqrt{n})}$

$= \lim\limits_{n\to\infty} \dfrac{-2}{(\sqrt{1+\frac{2}{n}}+\sqrt{1+\frac{1}{n}})(\sqrt{1+\frac{1}{n}}+1)(\sqrt{1+\frac{2}{n}}+1)} = \dfrac{-2}{2\times2\times2} = -\dfrac{1}{4}$

(4) 原式 $= \lim\limits_{n\to\infty} \dfrac{\frac{1-a^n}{1-a}}{\frac{1-b^n}{1-b}} = \dfrac{1-b}{1-a}$

(5) $\because 1-\frac{1}{k^2} = \frac{k^2-1}{k^2} = \frac{(k-1)(k+1)}{k^2} = \frac{k-1}{k}\cdot\frac{k+1}{k}$ $\therefore$ 原式 $= \frac{3}{2}\times\frac{1}{2}\times\frac{4}{3}\times\frac{2}{3}\times\cdots\times\frac{n}{n-1}\cdot\frac{n-2}{n-1}\cdot\frac{n+1}{n}\cdot\frac{n-1}{n} = \frac{n+1}{2n}$

$\therefore \lim\limits_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}$

(6) 由错位相减法. $\frac{1}{a}+\frac{2}{a^2}+\cdots+\frac{n}{a^n} = \frac{a-a^{2-n}}{(a-1)^2} - \frac{n}{a^n(a-1)}$

$\therefore$ 原式 $= \frac{a}{(a-1)^2}$.

南开大学 作业纸

系别________ 班级________ 姓名________ 第　　页

练习2.3

1. (1). 由习题1(4)-2. $n^{\frac{n}{2}} \le n! \le \left(\frac{n+1}{2}\right)^n$.

∴ $\left(\frac{1}{\sqrt{n}}\right)^n \le \frac{n!}{n^n} \le \left(\frac{1}{2}+\frac{1}{2n}\right)^n$.

其中 $\lim\limits_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2n}\right)^n = \lim\limits_{n\to\infty}\left(\frac{1}{2}\right)^n \cdot \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 0$

$\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{n}}\right)^n = 0$ 由两边夹定理. $\lim\limits_{n\to\infty}\frac{n!}{n^n}=0$

(2) $\left|\frac{a^n}{n!}\right| = \frac{|a|\cdot|a|\cdots\cdots|a|}{1\cdot 2\cdots\cdots n}$

对于数列 $\left\{\frac{|a|}{n}\right\}$. $n\to\infty$时 $\lim\limits_{n\to\infty}\frac{|a|}{n}=0$. ∴对 $\forall\varepsilon>0$. $\exists N$. 当 $n>N$ 时. $\left|\frac{a}{n}\right|<\varepsilon$.

取 $\varepsilon=\frac{1}{2}$. ∴ $\exists N_0$, $n>N_0$ 时. $\frac{|a|}{n}<\frac{1}{2}$.

∴ $\left|\frac{a^n}{n!}\right| < \frac{|a|^{N_0}}{N_0!}\cdot\left(\frac{1}{2}\right)^{n-N_0}\to 0$. ($n\to\infty$时) ∴ $\lim\limits_{n\to\infty}\frac{a^n}{n!}=0$.

(3) 当 $s\le 0$ 时. $\lim\limits_{n\to\infty}\frac{n^s}{a^n}=0$ 显然成立.

当 $s>0$ 时. 若 $n>[s]+1$

则 $\frac{n^s}{a^n} = \frac{n^s}{(a-1+1)^n} = \frac{n^s}{\sum\limits_{k=0}^{n}C_n^k(a-1)^k} < \frac{n^s}{C_n^{[s]+1}(a-1)^{[s]+1}} = \frac{n^s}{\frac{n(n-1)\cdots(n-[s])}{([s]+1)!}\cdot(a-1)^{[s]+1}}$

$< \frac{n^s}{(n-[s])^{[s]+1}\cdot\frac{(a-1)^{[s]+1}}{([s]+1)!}} = \frac{1}{(n-[s])^{1+[s]-s}\cdot\frac{(a-1)^{[s]+1}}{([s]+1)!}} \to 0$.

(4). 引理 由 $m^{\frac{1}{m}}<(1+m)^{\frac{1}{m}}<(2m)^{\frac{1}{m}}$. ∵ $\lim\limits_{\infty} m^{\frac{1}{m}} = \lim\limits_{\infty}(2m)^{\frac{1}{m}}=1$. ∴ $\lim\limits_{m\to\infty}(1+m)^{\frac{1}{m}}=1$.

不妨设 $a>1$. ∴ $\frac{1}{s}\cdot\frac{s\log_a n}{n^s} = \frac{1}{s}\cdot\frac{\log_a n^s}{n^s} < \frac{1}{s}\cdot\frac{\log_a([n^s]+1)}{[n^s]} = \frac{1}{s}\cdot\log_a\sqrt[[n^s]]{[n^s]+1}$.

由引理 $\sqrt[[n^s]]{[n^s]+1}\to 1$ ∴上式 $\to 0$. 又下式 ≥ 0 ∴ $\lim\limits_{n\to\infty}\frac{\log_a n}{n^s}=0$.

$0<a<1$ 时. 同理 ($\frac{1}{s}\cdot\frac{s\log_a n}{n^s} = \frac{1}{s}\cdot\frac{\ln\frac{1}{a}}{\ln a}\cdot\frac{s\log_{\frac{1}{a}} n}{n^s}$. 然后套这步骤.

2. (1) n为偶时.

$\because 0<$原式$=\frac{1}{n}+\left(-\frac{1}{2n}+\frac{1}{3n}\right)+\left(-\frac{1}{4n}+\frac{1}{5n}\right)+\cdots+\left(-\frac{1}{(n-2)n}+\frac{1}{(n-1)n}\right)-\frac{1}{n^2}<\frac{1}{n}\to 0$

$\therefore \lim\limits_{n\to\infty}$原式$=0$

n为奇时.

$0<$原式$=\frac{1}{n}+\left(-\frac{1}{2n}+\frac{1}{3n}\right)+\cdots+\left(-\frac{1}{(n-1)n}+\frac{1}{n^2}\right)<\frac{1}{n}\to 0$

$\therefore \lim\limits_{n\to\infty}$原式$=0$.

(2). $\sum\limits_{k=1}^{n}\frac{n+k}{n^2+k}>\sum\limits_{k=1}^{n}\frac{n+k}{n^2+n}=\frac{n^2+\frac{n^2+n}{2}}{n^2+n}=\frac{n}{n+1}+\frac{1}{2}\to\frac{3}{2}\quad(n\to\infty)$

$\sum\limits_{k=1}^{n}\frac{n+k}{n^2+k}<\sum\limits_{k=1}^{n}\frac{n+k}{n^2+1}=\frac{n^2+\frac{n^2+n}{2}}{n^2+1}=\frac{\frac{3}{2}(n^2+1)}{n^2+1}+\frac{\frac{1}{2}(n-3)}{n^2+1}\to\frac{3}{2}\quad(n\to\infty)$

$\therefore \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{n+k}{n^2+k}=\frac{3}{2}$.

(3). $\because (\sqrt{n})^n\le n!\le\left(\frac{n+1}{2}\right)^n$.

$\therefore n^{\frac{1}{2n}}\le(n!)^{\frac{1}{n^2}}\le\left(\frac{n+1}{2}\right)^{\frac{1}{n}}<n^{\frac{1}{n}}$ $\quad\because \lim\limits_{n\to\infty}\sqrt[n]{n}=1$ $\quad\therefore \lim\limits_{n\to\infty}n^{\frac{1}{2n}}=1$.

$\therefore \lim\limits_{n\to\infty}(n!)^{\frac{1}{n^2}}=1$

(4). $\because 2=(\sqrt[n]{2}-1+1)^n>1+n(\sqrt[n]{2}-1)>n(\sqrt[n]{2}-1)$

$\therefore 0<\sqrt[n]{2}-1<\frac{2}{n}$ $\quad\therefore 0<\log_2 n\cdot(\sqrt[n]{2}-1)<2\cdot\frac{\log_2 n}{n}$

由1-(4), $\lim\limits_{n\to\infty}\frac{2\log_2 n}{n}=0$ $\quad\therefore \lim\limits_{n\to\infty}\log_2 n\cdot(\sqrt[n]{2}-1)=0$

又 $|\cos n|<1$ $\quad\therefore \lim\limits_{n\to\infty}\log_2 n\cdot(\sqrt[n]{2}-1)\cdot\cos n=0$.

(5). $\because n<[n^{1+\frac{1}{n}}]<n^{1+\frac{1}{n}}$ $\quad\therefore 1<\frac{[n^{1+\frac{1}{n}}]}{n}<n^{\frac{1}{n}}$ $\quad\because \lim\limits_{n\to\infty}\sqrt[n]{n}=1$

$\therefore \lim\limits_{n\to\infty}\frac{[n^{1+\frac{1}{n}}]}{n}=1$.

3\. (1) 显然，$x_n>0$. $\because x_n^2=2x_{n-1} \Rightarrow (x_n+2)(x_n-2)=2(x_{n-1}-2) \Rightarrow x_n-2=\frac{2}{x_n+2}(x_{n-1}-2)$

$\because x_1=\sqrt{2}<2$. $\therefore x_2-2<0$ $\therefore x_3-2<0$ …… (归纳) $x_n-2<0$

$\therefore x_n-x_{n-1}=\sqrt{2x_{n-1}}-\sqrt{x_{n-1}}=\sqrt{x_{n-1}}(\sqrt{2}-\sqrt{x_{n-1}})>0$. 即$\{x_n\}$递增. 且$x_n<2$.

$\therefore$ 由单调收敛定理 $\{x_n\}$收敛.

(2) $x_n-x_{n-1}=\frac{1}{\sqrt{n}}-2(\sqrt{n}-\sqrt{n-1})=\frac{1}{\sqrt{n}}-\frac{2}{\sqrt{n}+\sqrt{n-1}}=\frac{\sqrt{n-1}-\sqrt{n}}{\sqrt{n}(\sqrt{n}+\sqrt{n-1})}<0$.

又$x_n-x_{n-1}=\frac{1}{\sqrt{n}}-\frac{2}{\sqrt{n}+\sqrt{n-1}}>\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n-1}}$.

$\therefore \forall n$, $x_n=x_1+\sum_{k=2}^{n}(x_k-x_{k-1})>x_1+\sum_{k=2}^{n}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k-1}}\right)=-1+\frac{1}{\sqrt{n}}-1=-2+\frac{1}{\sqrt{n}}>-2$.

$\therefore$ 由单调收敛定理 $\{x_n\}$收敛.

4\. $|x_n-x_m|=\frac{\sin(m+1)}{2^{m+1}}+\frac{\sin(m+2)}{2^{m+2}}+\cdots+\frac{\sin n}{2^n}$ (由于$m>4$时$2^m>m^2$).

$<\frac{\sin(m+1)}{(m+1)^2}+\frac{\sin(m+2)}{(m+2)^2}+\cdots+\frac{\sin n}{n^2}$ (由 $\sin x<1<\frac{x}{x-1}$).

$<\frac{1}{m(m+1)}+\frac{1}{(m+1)(m+2)}+\cdots+\frac{1}{(n-1)n}$

$=\frac{1}{m}-\frac{1}{m+1}+\frac{1}{m+1}-\frac{1}{m+2}+\cdots+\frac{1}{n-1}-\frac{1}{n}=\frac{1}{m}-\frac{1}{n}<\frac{1}{m}$.

令$\frac{1}{m}<\varepsilon$. $\therefore m>\frac{1}{\varepsilon}$.

$\therefore$对$\forall\varepsilon>0$. 取$N=\max\{[\frac{1}{\varepsilon}]+1, 5\}$.

则$n>m>N$时. $|x_n-x_m|<\frac{1}{m}<\varepsilon$. 由柯西收敛准则 $\{x_n\}$收敛.

练习2.4

1. (1). $\forall \varepsilon>0$. $\forall X=\frac{\ln\varepsilon}{\ln q}$ 当 $x>X$ 时. $x>\frac{\ln\varepsilon}{\ln q}$ $\Rightarrow |q^x|<\varepsilon$ $\therefore \lim\limits_{x\to+\infty} q^x=0$.

(2). $\forall\varepsilon>0$. $\exists\delta=\begin{cases}\min\{a^{\varepsilon}-1,\ 1-a^{-\varepsilon}\} & a>1\\ \min\{1-a^{\varepsilon},\ a^{-\varepsilon}-1\} & 0<a<1.\end{cases}$ $0<|x-1|<\delta$ 时, 有 $|\log_a x|<\varepsilon$ 成立

$\therefore \lim\limits_{x\to1}\log_a x=0$.

(3). $\because \left|\frac{5(x+1)}{5(2x^2-3)}-\frac{3}{5}\right|=\left|\frac{14+5x-6x^2}{5(2x^2-3)}\right|=\frac{|x-2|\cdot|6x+7|}{5|2x^2-3|}$

当 $0<|x-2|<\frac{1}{2}$ 即 $\frac{3}{2}<x<\frac{5}{2}$ 时. $|6x+7|<22$, $|2x^2-3|>\frac{3}{2}$. $\therefore$ 上式 $<\frac{|x-2|\cdot22}{5\cdot\frac{3}{2}}=\frac{44}{15}|x-2|$.

$\therefore$ 对 $\forall\varepsilon>0$. $\exists\delta=\min\{\frac{1}{2},\ \frac{15}{44}\varepsilon\}$. $0<|x-2|<\delta$ 时. 有 $\left|\frac{x+1}{2x^2-3}-\frac{3}{5}\right|<\varepsilon$ $\therefore \lim\limits_{x\to2}\frac{x+1}{2x^2-3}=\frac{3}{5}$

(4) $\because \left|\frac{x^2-9}{x^2+x-12}-\frac{6}{7}\right|=\frac{(x-3)^2}{7|(x+4)(x-3)|}=\frac{|x-3|}{7|x+4|}$ 当 $0<|x-3|<1$ 即 $2<x<4$ 时. $|x+4|>6$

$\therefore$ 上式 $<\frac{|x-3|}{42}$ $\therefore$ 对 $\forall\varepsilon>0$. $\exists\delta=\min\{1,\ 42\varepsilon\}$. 当 $0<|x-3|<\delta$ 时 $\left|\frac{x^2-9}{x^2+x-12}-\frac{6}{7}\right|<\varepsilon$. $\lim\limits_{x\to3}\frac{x^2-9}{x^2+x-12}=\frac{6}{7}$

(5) 当 $-\delta<x<0$ 时. 令 $\left|\arctan\frac{1}{x}+\frac{\pi}{2}\right|<\varepsilon$. $\therefore \arctan\frac{1}{x}<\varepsilon-\frac{\pi}{2}$. $(0<\varepsilon<\frac{\pi}{2})$.

$\therefore \frac{1}{x}<\tan(\varepsilon-\frac{\pi}{2})$ $\therefore x>\frac{1}{\tan(\varepsilon-\frac{\pi}{2})}$ 取 $\delta=-\frac{1}{\tan(\varepsilon-\frac{\pi}{2})}$

$\therefore$ 对 $\forall\varepsilon>0$. $\exists\delta>0$ $-\delta<x<0$ 时. $\left|\arctan\frac{1}{x}+\frac{\pi}{2}\right|<\varepsilon$ $\therefore \lim\limits_{x\to0^-}\arctan\frac{1}{x}=-\frac{\pi}{2}$

(6). 当 $x_0=0$ 时容易证明. 当 $x_0>0$ 时.

$|\sqrt[3]{x}-\sqrt[3]{x_0}|=\frac{|x-x_0|}{|x^{\frac{2}{3}}+x^{\frac{1}{3}}x_0^{\frac{1}{3}}+x_0^{\frac{2}{3}}|}<\frac{|x-x_0|}{x_0^{\frac{2}{3}}}$ $\therefore$ 对 $\forall\varepsilon>0$. $\exists\delta=x_0^{\frac{2}{3}}\varepsilon$.

$0<|x-x_0|<\delta$ 时. $|\sqrt[3]{x}-\sqrt[3]{x_0}|<\varepsilon$ $\therefore \lim\limits_{x\to x_0}\sqrt[3]{x}=\sqrt[3]{x_0}$. ($x_0<0$ 时同理).

(7) 令 $|e^x-e^{x_0}|<\varepsilon$ 即 $e^{x_0}-\varepsilon<e^x<e^{x_0}+\varepsilon$. (不妨设 $\varepsilon<e^{x_0}$)

$\therefore \ln(e^{x_0}-\varepsilon)<x<\ln(e^{x_0}+\varepsilon)$ 取 $\delta=\min\{\ln(e^{x_0}+\varepsilon)-x_0,\ x_0-\ln(e^{x_0}-\varepsilon)\}$

$\therefore 0<|x-x_0|<\delta$ 时 $\forall\varepsilon$. $|e^x-e^{x_0}|<\varepsilon$ $\therefore \lim\limits_{x\to x_0}e^x=e^{x_0}$.

2. (1) $\frac{1}{n+1}<x_0<\frac{1}{n-1}$.

当 $x\in(\frac{1}{n+1},x_0)$ 时. $\frac{1}{x}\in(n,n+1)$ $f(x)=\frac{1}{x}-n$ $\therefore \lim\limits_{x\to x_0^-}f(x)=\lim\limits_{x\to x_0^-}(\frac{1}{x}-n)=0$

当 $x\in(x_0,\frac{1}{n-1})$ 时 $\frac{1}{x}\in(n-1,n)$ $f(x)=\frac{1}{x}-n+1$. $\therefore \lim\limits_{x\to x_0^+}f(x)=\lim\limits_{x\to x_0^+}(\frac{1}{x}-n+1)=1$.

(2). $x\in(\frac{1}{2},1)$ 时 $f(x)=\ln(x+2)$ $\therefore \lim\limits_{x\to 1^-}f(x)=\ln 3$

$x\in(1,\frac{\pi}{2})$ 时 $f(x)=-\sin x$ $\therefore \lim\limits_{x\to 1^+}f(x)=-\sin 1$

3. (1). $\exists\varepsilon_0>0$. $\forall\delta>0$. $\exists x_1$ 满足 $0<x_1-x_0<\delta$. 使 $|f(x_1)-A|\geq\varepsilon_0$.

(2). $\exists\varepsilon_0>0$ $\forall X>0$. $\exists x_1$ 满足 $|x_1|>X$ 使 $|f(x_1)-A|\geq\varepsilon_0$.

练习2.5. 1. (1) $\lim\limits_{x\to1}\frac{x^m-1}{x^n-1}=\lim\limits_{x\to1}\frac{(x-1)(1+x+\cdots+x^{m-1})}{(x-1)(1+x+\cdots+x^{n-1})}=\lim\limits_{x\to1}\frac{1+x+\cdots+x^{m-1}}{1+x+\cdots+x^{n-1}}=\frac{m}{n}$

(2) $\lim\limits_{x\to1}\frac{\sqrt[m]{x}-1}{\sqrt[n]{x}-1}=\lim\limits_{x\to1}\frac{\frac{x-1}{1+x^{\frac{1}{m}}+x^{\frac{2}{m}}+\cdots+x^{\frac{m-1}{m}}}}{\frac{x-1}{1+x^{\frac{1}{n}}+x^{\frac{2}{n}}+\cdots+x^{\frac{n-1}{n}}}}=\frac{n}{m}$

(3) $\lim\limits_{x\to-8}\frac{\sqrt{1-x}-3}{2+\sqrt[3]{x}}=\lim\limits_{x\to-8}\frac{(\sqrt{1-x}-3)(\sqrt{1-x}+3)(4+x^{\frac{2}{3}}-2x^{\frac{1}{3}})}{(\sqrt{1-x}+3)(2+\sqrt[3]{x})(4+x^{\frac{2}{3}}-2x^{\frac{1}{3}})}=\lim\limits_{x\to-8}\frac{-x-8}{x+8}\cdot\frac{4+x^{\frac{2}{3}}-2x^{\frac{1}{3}}}{\sqrt{1-x}+3}$

$=-1\cdot\frac{4+4+4}{6}=-2$

(4) $\because\ \frac{\sqrt[m]{1+\alpha x}-1}{x}=\frac{\frac{1+\alpha x-1}{1+(1+\alpha x)^{\frac{1}{m}}+(1+\alpha x)^{\frac{2}{m}}+\cdots+(1+\alpha x)^{\frac{m-1}{m}}}}{x}$

$\therefore\ \lim\limits_{x\to0}\frac{\sqrt[m]{1+\alpha x}-1}{x}=\frac{\alpha}{m}$ 同理 $\lim\limits_{x\to0}\frac{\sqrt[n]{1+\beta x}-1}{x}=\frac{\beta}{n}$

$\therefore\ \lim\limits_{x\to0}\frac{\sqrt[m]{1+\alpha x}\sqrt[n]{1+\beta x}-1}{x}=\lim\limits_{x\to0}\left[\frac{\sqrt[m]{1+\alpha x}-1}{x}+\sqrt[m]{1+\alpha x}\cdot\frac{\sqrt[n]{1+\beta x}-1}{x}\right]=\frac{\alpha}{m}+\frac{\beta}{n}$.

(5) $\lim\limits_{x\to+\infty}\frac{\ln(2+e^{3x})}{\ln(3+e^{2x})}=\lim\limits_{x\to+\infty}\frac{\ln e^{3x}(2e^{-3x}+1)}{\ln e^{2x}(3e^{-2x}+1)}=\lim\limits_{x\to+\infty}\frac{3x+\ln(1+2\cdot e^{-3x})}{2x+\ln(1+3\cdot e^{-2x})}=\lim\limits_{x\to+\infty}\frac{3+\frac{\ln(1+2\cdot e^{-3x})}{x}}{2+\frac{\ln(1+3\cdot e^{-2x})}{x}}$

$\because\ \ln(1+\frac{2}{e^{3x}})\to0\quad\therefore\ \frac{\ln(1+\frac{2}{e^{3x}})}{x}\to0$ 同理 $\frac{\ln(1+\frac{3}{e^{2x}})}{x}\to0\quad\therefore$ 上式 $=\frac{3}{2}$.

(6) $\lim\limits_{x\to\pi}\frac{\sin mx}{\sin nx}=\lim\limits_{x\to\pi}\frac{\sin[m(x-\pi)+m\pi]}{\sin[n(x-\pi)+n\pi]}=\lim\limits_{x\to\pi}\frac{\cos m\pi}{\cos n\pi}\cdot\frac{\sin m(x-\pi)}{\sin n(x-\pi)}$.

$=\lim\limits_{x\to\pi}\frac{\cos m\pi}{\cos n\pi}\cdot\frac{m(x-\pi)}{n(x-\pi)}=\frac{m}{n}\cdot\frac{\cos m\pi}{\cos n\pi}=\pm\frac{m}{n}$.

练习2.6

1.(1) $\because \lim\limits_{n\to\infty} n^{\frac{1}{n}}=1$. $\therefore \lim\limits_{x\to+\infty}[x]^{\frac{1}{[x]}}=\lim\limits_{x\to+\infty}([x]+1)^{\frac{1}{[x]+1}}=1$

$\because ([x])^{\frac{1}{[x]}}>x^{\frac{1}{x}}>([x]+1)^{\frac{1}{[x]+1}}$ $\therefore \lim\limits_{x\to+\infty}x^{\frac{1}{x}}=1$.

(2) $0<\frac{x^s}{a^x}<\frac{x^s}{x^{s+1}}=\frac{1}{x}\to 0$ $\therefore \lim\limits_{x\to+\infty}\frac{x^s}{a^x}=0$

(3) 记$y=\log_a x$. 不妨设$a>1$. $\therefore x\to+\infty$时$y\to+\infty$ $\therefore x=a^y$

$\therefore \lim\limits_{x\to+\infty}\frac{\log_a x}{x^s}=\frac{1}{s}\lim\limits_{ys\to+\infty}\frac{ys}{a^{ys}}$ 由(2) $\lim\limits_{x\to+\infty}\frac{x}{a^x}=0$ $\therefore$ 上式$=0$.($0<a<1$时同理)

(4). 记$y=\frac{1}{x}$. 则$x\to0^+$时 $y\to+\infty$

$\therefore \lim\limits_{x\to0^+}x^s\log_a x=-\lim\limits_{y\to+\infty}(\frac{1}{y})^s\cdot\log_a y=-\lim\limits_{y\to+\infty}\frac{\log_a y}{y^s}=0$ (由(3)).

2.(1). $\because \frac{1}{x}-1<[\frac{1}{x}]<\frac{1}{x}$ $\Rightarrow 1-x<x\cdot[\frac{1}{x}]<1$ $\therefore \lim\limits_{x\to0}x[\frac{1}{x}]=1$.

(2) $\lim\limits_{n\to\infty}\sin(\pi\sqrt{n^2+1})=\lim\limits_{n\to\infty}\sin(\pi(\sqrt{n^2+1}-n)+n\pi)=\lim\limits_{n\to\infty}\cos n\pi\cdot\sin\pi(\sqrt{n^2+1}-n)$

$\because |\cos n\pi|\le1$, 而$\lim\limits_{n\to\infty}(\sqrt{n^2+1}-n)=0\Rightarrow\lim\limits_{n\to\infty}\sin\pi(\sqrt{n^2+1}-n)=0$. $\therefore \lim\limits_{n\to\infty}\sin(\pi\sqrt{n^2+1})=0$

(3) 原式$=\lim\limits_{n\to\infty}\frac{1}{\sin\frac{x}{2^n}}\cdot\cos\frac{x}{2}\cdot\cos\frac{x}{4}\cdots\cos\frac{x}{2^n}\cdot\sin\frac{x}{2^n}=\lim\limits_{n\to\infty}\frac{1}{\sin\frac{x}{2^n}}\cdot\cos\frac{x}{2}\cos\frac{x}{4}\cdots\cos\frac{x}{2^{n-1}}\cdot\frac{1}{2}\sin\frac{x}{2^{n-1}}$

$=\cdots=\lim\limits_{n\to\infty}\frac{1}{2^n}\cdot\frac{\sin x}{\sin\frac{x}{2^n}}=\frac{1}{2^n}\lim\limits_{n\to\infty}\frac{\sin x}{\frac{x}{2^n}}=\frac{\sin x}{x}$

3. (1) $\lim\limits_{x\to x_0^+}f(x)=A \Leftrightarrow$ 对任意满足①$x_n\to x_0$ ② $\forall x_n>x_0$的$\{x_n\}$. 均有$\lim\limits_{n\to\infty}f(x_n)=A$.

(2) $\lim\limits_{x\to-\infty}f(x)=A \Leftrightarrow$ 对任意满足$x_n\to-\infty$的数列$\{x_n\}$均有$\lim\limits_{n\to\infty}f(x_n)=A$.

4. (1) $\lim\limits_{x\to x_0^-}f(x)$存在 $\Leftrightarrow$ 对$\forall\varepsilon>0$. $\exists\delta>0$. 当$x_0-\delta<x, x'<x_0$时 $|f(x)-f(x')|<\varepsilon$

(2) $\lim\limits_{x\to\infty}f(x)$存在 $\Leftrightarrow$ 对$\forall\varepsilon>0$ $\exists M>0$. 当$|x|>M$, $|x'|>M$时 $|f(x)-f(x')|<\varepsilon$

练习2.7

1. (1) 当 $n>2$ 时. 令 $\left|\frac{n^2-3}{2n+1}\right|>M$. $\because \left|\frac{n^2-3}{2n+1}\right|>\left|\frac{n^2-4}{2n+4}\right|=\frac{n-2}{2}>M$ $\therefore n>2M+2$

$\therefore$ 对 $\forall M>0$. 取 $N=2M+2$. 当 $n>N=2M+2$ 时.

有 $M<\frac{n-2}{2}=\frac{n^2-4}{2n+4}<\frac{n^2-3}{2n+1}$ $\therefore \left\{\frac{n^2-3}{2n+1}\right\}$ 为无穷大量.

(2) $\because \frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots+\left(\frac{1}{2^k+1}+\frac{1}{2^k+2}+\cdots+\frac{1}{2^{k+1}}\right)$

$>\frac{1}{2}+\frac{1}{4}\times 2+\frac{1}{8}\times 4+\cdots+\frac{1}{2^{k+1}}\times 2^k=\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}=\frac{k+1}{2}$

对 $\forall M>0$. 取 $N=2^{[2M]+1}$ $n>N$ 时.

$1+\frac{1}{2}+\cdots+\frac{1}{n}>\frac{1}{2}+\frac{1}{4}\times 2+\frac{1}{8}\times 3+\cdots+\frac{1}{2^{[2M]+1}}\times 2^{[2M]}=\frac{[2M]+1}{2}>M$

$\therefore \left\{1+\frac{1}{2}+\cdots+\frac{1}{n}\right\}$ 为无穷大量.

2. (1) $\exists M_0>0$. $\forall \delta>0$, $\exists x'\in(x_0-\delta, x_0)$ 使 $f(x')\le M_0$.

(2) $\exists M_0>0$. $\forall X>0$. $\exists x'\in(X,+\infty)$ 使 $f(x')\ge -M_0$

(3) $\exists M_0>0$. $\forall X>0$. $\exists x'$ 满足 $|x'|>X$ 使 $f(x')\le M_0$.

3. (1) $\forall M>0$. 取 $\delta=a^M>0$. 当 $0<x<\delta$ 时. $\log_a x>M$ $\therefore \lim\limits_{x\to 0^+}\log_a x=+\infty$ $(0<a<1)$

(2) $\forall M>0$. 取 $X=M>0$. $x<-X=-M<0$ 时. $\frac{x^2+x}{x-1}<\frac{x^2-x}{x-1}=x<-M$ $\therefore \lim\limits_{x\to-\infty}\frac{x^2+x}{x-1}=-\infty$

(3) $\forall M>0$ 取 $X=\arctan\frac{1}{M}>0$. $|x|>X=\arctan\frac{1}{M}$ 时. $\left|\cot\frac{1}{x}\right|>M$ $\therefore \lim\limits_{x\to\infty}\cot\frac{1}{x}=\infty$.

4. (1) $\lim\limits_{x\to x_0}f(x)=+\infty \Leftrightarrow$ 对任意满足: ① $x_n\to x_0$ ② $x_n\ne x_0$ 的 $\{x_n\}$, 均有 $\lim\limits_{n\to\infty}f(x_n)=+\infty$

(2) $\lim\limits_{x\to\infty}f(x)=-\infty \Leftrightarrow$ 对任意满足 $x_n\to\infty$ 的数列 $\{x_n\}$ 均有 $\lim\limits_{n\to\infty}f(x_n)=-\infty$.

5. (1) $\lim\limits_{x\to0}\dfrac{\log_a(1+x)}{x}=\lim\limits_{x\to0}\log_a(1+x)^{\frac{1}{x}}=\lim\limits_{x\to\infty}\log_a(1+\frac{1}{x})^{x}=\log_a e=\dfrac{1}{\ln a}$

(2) $\lim\limits_{x\to0}\dfrac{a^x-1}{x}=\lim\limits_{x\to0}\dfrac{e^{x\ln a}-1}{x}=\lim\limits_{x\to0}\dfrac{x\ln a}{x}=\ln a$.

(3) $\lim\limits_{x\to a}\dfrac{a^x-x^a}{x-a}\overset{t=x-a}{=\!=\!=}\lim\limits_{t\to0}\dfrac{a^a\cdot a^t-(a+t)^a}{t}=\lim\limits_{t\to0}\dfrac{a^a(a^t-1)}{t}+\lim\limits_{t\to0}\dfrac{a^a\left[1-(1+\frac{t}{a})^a\right]}{t}$

$\because\lim\limits_{t\to0}\dfrac{a^t-1}{t}=\ln a$ (由(2)) $\therefore\lim\limits_{t\to0}\dfrac{1-(1+\frac{t}{a})^a}{t}=\lim\limits_{t\to0}\dfrac{1-\left[(1+\frac{t}{a})^{\frac{a}{t}}\right]^t}{t}=\lim\limits_{t\to0}\dfrac{1-e^t}{t}=-1$

$\therefore\lim\limits_{x\to a}\dfrac{a^x-x^a}{x-a}=a^a(\ln a-1)$

(4) $\lim\limits_{x\to0}\dfrac{\ln(e^x+\sin^2x)-x}{\ln(e^{2x}+\arcsin^2x)-2x}=\lim\limits_{x\to0}\dfrac{\ln\left(\frac{e^x+\sin^2x}{e^x}\right)}{\ln\left(\frac{e^{2x}+\arcsin^2x}{e^{2x}}\right)}=\lim\limits_{x\to0}\dfrac{\frac{\sin^2x}{e^x}}{\frac{\arcsin^2x}{e^{2x}}}=\lim\limits_{x\to0}\dfrac{e^x\cdot\sin^2x}{\arcsin^2x}$

$=\lim\limits_{x\to0}e^x\cdot\left(\dfrac{x}{\arcsin x}\right)^2=\lim\limits_{x\to0}e^x\cdot\left(\dfrac{\sin x}{x}\right)^2=1$

(5) $\lim\limits_{x\to0}\dfrac{\ln\tan(\frac{\pi}{4}+ax)}{\sin bx}=\lim\limits_{x\to0}\dfrac{\ln\left(1+\frac{2\tan(ax)}{1-\tan(ax)}\right)}{\sin bx}=\lim\limits_{x\to0}\dfrac{\frac{2\tan ax}{1-\tan ax}}{\sin bx}$.

$=\lim\limits_{x\to0}\dfrac{2\tan ax}{\sin bx(1-\tan ax)}=\lim\limits_{x\to0}\dfrac{2a}{b(1-\tan ax)}=\dfrac{2a}{b}$.

(6) $\lim\limits_{x\to0}\dfrac{(1+x)^\alpha-1}{x}=\lim\limits_{x\to0}\dfrac{e^{\alpha\ln(1+x)}-1}{x}=\lim\limits_{x\to0}\dfrac{\alpha\ln(1+x)}{x}=\alpha\cdot\lim\limits_{x\to0}\dfrac{x}{x}=\alpha$.

(7) $\lim\limits_{n\to\infty}\sqrt{n}(\sqrt[n]{n}-1)=\lim\limits_{x\to+\infty}\sqrt{x}(x^{\frac{1}{x}}-1)=\lim\limits_{x\to+\infty}\sqrt{x}(e^{\frac{\ln x}{x}}-1)=\lim\limits_{x\to+\infty}\dfrac{e^{\frac{\ln x}{x}}-1}{\frac{\ln x}{x}}\cdot\sqrt{x}\cdot\dfrac{\ln x}{x}$

$=\lim\limits_{y\to0}\dfrac{e^y-1}{y}\cdot\lim\limits_{x\to+\infty}\dfrac{\ln x}{\sqrt{x}}=1\cdot0=0$

(8) $\because\dfrac{\sin x}{x}\sim\dfrac{\tan x}{x}\cdot\cos x\sim\cos x\sim1-\dfrac{x^2}{2}\quad\therefore1-\dfrac{\sin x}{x}\sim\dfrac{x^2}{2}$

$\therefore\lim\limits_{x\to0}\ln\left(\dfrac{\sin x}{x}\right)\ln\dfrac{1}{x}=$ 原式 $=\lim\limits_{x\to0}\ln\dfrac{1}{x}\cdot\ln\left(\dfrac{\sin x}{x}-1+1\right)=\lim\limits_{x\to0}\ln\dfrac{1}{x}\cdot\left(\dfrac{\sin x}{x}-1\right)$

$=\lim\limits_{x\to0}\ln x\cdot\dfrac{x^2}{2}=\lim\limits_{x\to0}\dfrac{\ln x^{x^2}}{2}=0$

南開大學 作业纸

系别______ 班级______ 姓名______ 第 页

习题2：

1. $\because a<1$ $\therefore \lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right|=a=\frac{a+a}{2}<\frac{a+1}{2}<1$.

- 由极限保号性，$\exists N_1\in N^*$，$n\geq N_1$时，$\left|\frac{x_{n+1}}{x_n}\right|<\frac{a+1}{2}$.
- $\because \frac{a+1}{2}<1$ $\therefore \lim_{n\to\infty}\left(\frac{a+1}{2}\right)^{n-N_1}=0$ $\therefore \exists N_2\in N^*$，$n\geq N_2$时 $\left(\frac{a+1}{2}\right)^{n-N_1}<\varepsilon$.

$\Rightarrow$ 对 $\forall\varepsilon>0$，$\exists N=\max\{N_1,N_2\}$，$n>N$时，

$$|x_n|=\left|\frac{x_n}{x_{n-1}}\right|\cdot\left|\frac{x_{n-1}}{x_{n-2}}\right|\cdots\left|\frac{x_{N_1+1}}{x_{N_1}}\right|\cdot|x_{N_1}|<\left(\frac{1+a}{2}\right)^{n-N_1}\cdot|x_{N_1}|<|x_{N_1}\cdot\varepsilon|.$$

$\therefore \lim_{n\to\infty}x_n=0$

2. 递推式特征方程为 $2x^2=x+1$，$x_1=1$，$x_2=-\frac{1}{2}$

$\therefore x_n=A+B\cdot\left(-\frac{1}{2}\right)^{n-1}$ $\therefore \begin{cases}x_1=A+B=a\\ x_2=A-\frac{B}{2}=b\end{cases}\Rightarrow\begin{cases}A=\frac{a+2b}{3}\\ B=\frac{2a-2b}{3}\end{cases}$

$\therefore x_n=\frac{a+2b}{3}+\frac{2(a-b)}{3}\cdot\left(-\frac{1}{2}\right)^{n-1}$

显然 $\{x_n\}$ 收敛 且 $\lim_{n\to\infty}x_n=\frac{a+2b}{3}$.

3. $\frac{a_n}{n}=\frac{1}{n}\left(\sum_{i=1}^{n}a_i-\sum_{i=1}^{n-1}a_i\right)$

$=\frac{1}{n}\sum_{i=1}^{n}a_i-\frac{n-1}{n}\cdot\frac{1}{n-1}\sum_{i=1}^{n}a_i\to A-\frac{n-1}{n}A=\frac{1}{n}A\to0$.

$\lim_{n\to\infty}\frac{a_n}{n}=0$.

南開大學 作业纸

系别________ 班级________ 姓名__________ 第　　页

4. (引理) 若 $\lim_{n\to\infty} x_n = a$，则 $\lim_{n\to\infty} \frac{1}{n}(x_1+\cdots+x_n)=a$． 证明见课本P29 例5.

证：∵ $x_n>0$　∴ $a\geq 0$.

①当 $a=0$ 时．$0<\sqrt[n]{x_1\cdots x_n}\leq \frac{x_1+\cdots+x_n}{n}\longrightarrow a$（由引理）而 $a=0$．

由两边夹定理．$\sqrt[n]{x_1\cdots x_n}\to a=0$．

②当 $a\neq 0$ 时． 由 $\lim_{n\to\infty} x_n=a \Rightarrow \lim_{n\to\infty}\ln x_n=\ln a$．

由引理，∴ $\ln\sqrt[n]{x_1\cdots x_n}=\frac{1}{n}\sum_{i=1}^{n}\ln x_i\to\ln a$，∴ $\sqrt[n]{x_1\cdots x_n}\to e^{\ln a}=a$．

逆命题为假．反例如：$x_n=\begin{cases}2 & n为奇\\ \frac{1}{2} & n为偶\end{cases}$．

5．由习题1-4．知 $\lim_{n\to\infty}\sqrt[n]{\frac{x_2}{x_1}\cdots\cdots\frac{x_{n+1}}{x_n}}=\lim_{n\to\infty}\sqrt[n]{\frac{x_{n+1}}{x_1}}=a$．

∵ $\lim_{n\to\infty}\sqrt[n]{x_1}=1$．　∴ $\lim_{n\to\infty}\sqrt[n]{x_{n+1}}=a$　∴ $\lim_{n\to\infty}\sqrt[n]{x_n}=a$．

逆命题为假． 反例如：$x_n=(-1)^n+2$，∵ $1\leq x_n\leq 3$　∴ $\lim_{n\to\infty}\sqrt[n]{x_n}=1$．

而 $\frac{x_{n+1}}{x_n}=\begin{cases}\frac{1}{3} & n为偶\\ 3 & n为奇\end{cases}$，$\lim_{n\to\infty}\frac{x_{n+1}}{x_n}$ 不存在．

6. $\because \{x_n\}$收敛 $\therefore \{x_n\}$有界. 设$|x_n| \le M$.

$$\therefore \left|\frac{x_1y_n+x_2y_{n-1}+\cdots+x_ny_1}{n}-ab\right| = \left|\frac{(x_1y_n-ab)+(x_2y_{n-1}-ab)+\cdots+(x_ny_1-ab)}{n}\right|.$$

$$< \frac{|x_1y_n-ab|+\cdots+|x_ny_1-ab|}{n} = \frac{|x_1y_n-x_1b+x_1b-ab|+\cdots\cdots+|x_ny_1-x_nb+x_nb-ab|}{n}$$

$$< \frac{|x_1y_n-x_1b|+|x_1b-ab|+\cdots\cdots+|x_ny_1-x_nb|+|x_nb-ab|}{n}$$

$\because \lim\limits_{n\to\infty} x_n=a$, $\lim\limits_{n\to\infty} y_n=b$. $\therefore$对$\forall\varepsilon>0$. $\exists N_1, N_2 \in N^*$.

$n>N_1$时. 有$|x_n-a|<\varepsilon$. $n>N_2$时. 有$|y_n-b|<\varepsilon$.

取$N=\max\{N_1,N_2\}$ $\therefore n>N$时. 对$\forall\varepsilon>0$.

$$\therefore 上式 = \frac{|x_1|\cdot|y_n-b|+|x_2|\cdot|y_{n-1}-b|+\cdots+|x_{n-N}|\cdot|y_{N+1}-b|+|x_{n-N+1}|\cdot|y_N-b|+\cdots+|x_n|\cdot|y_1-b|}{n}$$

$$+\frac{|b|}{n}\cdot\left[|x_1-a|+\cdots+|x_N-a|+|x_{N+1}-a|+\cdots+|x_n-a|\right]$$

$$< \frac{M}{n}\left[(n-N)\varepsilon+\sum_{i=1}^{N}|y_i-b|\right]+\frac{|b|}{n}\cdot\left[(n-N)\varepsilon+\sum_{i=1}^{N}|x_i-a|\right]$$

$$\triangleq \frac{(M+|b|)}{n}(n-N)\varepsilon+\frac{S}{n} \to \frac{(M+|b|)}{n}(n-N)\varepsilon < (M+|b|-1)\varepsilon.$$

$$\therefore \lim_{n\to\infty}\frac{1}{n}(x_1y_n+\cdots+x_ny_1)=ab.$$

7. 令 $b_n=a_n-a$. 只需证 $\lim\limits_{n\to\infty}b_n=0$. 不妨设 $\{a_n\}$ 单增. 则 $\{b_n\}$ 单增.

假设 $\exists b_i$, $b_i>0$.

$\therefore \dfrac{b_1+\cdots+b_n}{n}=\dfrac{b_1+\cdots+b_{i-1}}{n}+\dfrac{b_i+\cdots+b_n}{n}>\dfrac{b_1+\cdots+b_{i-1}}{n}+\dfrac{n-i}{n}b_i$ 由极限保序性,

$\therefore \lim\limits_{n\to\infty}\dfrac{b_1+\cdots+b_n}{n}>\lim\limits_{n\to\infty}\left(\dfrac{b_1+\cdots+b_{i-1}}{n}+\dfrac{n-i}{n}b_i\right)=\lim\limits_{n\to\infty}\left(\dfrac{n-i}{n}b_i\right)=b_i>0$.

与 $\lim\limits_{n\to\infty}\dfrac{b_1+\cdots+b_n}{n}=0$ 矛盾. $\therefore$ 对 $\forall b_i$, $b_i\le 0$.

$\therefore \dfrac{b_1+\cdots+b_n}{n}\le\dfrac{nb_n}{n}b_n=b_n\le 0$. $\because \lim\limits_{n\to\infty}\dfrac{b_1+\cdots+b_n}{n}=0$. 由两边夹定理. $\lim\limits_{n\to\infty}b_n=0$.

8. $\because x_{n+1}=\sqrt{x_ny_n}\le\dfrac{x_n+y_n}{2}=y_{n+1}$. $\therefore \begin{cases} x_{n+1}=\sqrt{x_ny_n}\ge\sqrt{x_n^2}=x_n & x_n\uparrow \\ y_{n+1}=\dfrac{x_n+y_n}{2}\le\dfrac{2y_n}{2}=y_n. & y_n\downarrow \end{cases}$

$\therefore x_1\le x_n\le x_{n+1}\le y_{n+1}\le y_n\le y_1$ 即: $\{x_n\}$ 递增且以 y_1 为上界, $\{y_n\}$ 递减且以 x_1 为下界.

由单调收敛定理. $\{x_n\}$、$\{y_n\}$ 极限均存在. 设 $\lim\limits_{n\to\infty}x_n=A$, $\lim\limits_{n\to\infty}y_n=B$.

由保序性 $A\le B\le A$. $\therefore A=B$.

9. $x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)\ge\frac{1}{2}\cdot 2\sqrt{a}=\sqrt{a}$. 且 $x_{n+1}-x_n=\frac{1}{2}\left(\frac{a}{x_n}-x_n\right)=\dfrac{a-x_n^2}{2x_n}\le 0$.

$\therefore \{x_n\}$ 递减且有下界. 由单调收敛定理 $\{x_n\}$ 收敛.

设 $\lim\limits_{n\to\infty}x_n=A$. 对 $x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)$ 两边取极限有:

$A=\frac{1}{2}\left(A+\frac{a}{A}\right)\Rightarrow A^2=a$. $\because x_n>0$ $\therefore A>0$. $\therefore A=\sqrt{a}$. 即 $\lim\limits_{n\to\infty}x_n=\sqrt{a}$

10. $\because A_{n+1}-A_n=|a_{n+2}-a_{n+1}|>0$. $\therefore \{A_n\}$递增. 又$\because$其有界 $\therefore \{A_n\}$收敛

由柯西收敛原理. 对$\forall \varepsilon>0$. $\exists N$, $n>N$时 $|A_{n+1}-A_n|=|a_{n+1}-a_n|<\varepsilon$.

由柯西收敛原理. $\{a_n\}$收敛.

11. $\because |x_{n+2}-x_{n+1}|\le\alpha|x_{n+1}-x_n|\le\alpha^2|x_n-x_{n-1}|\le\cdots\le\alpha^n|x_2-x_1|$. 即 $|x_{n+1}-x_n|\le\alpha^{n-1}|x_2-x_1|$.

对任意的$m>n$.

$|x_m-x_n|\le|x_m-x_{m-1}|+|x_{m-1}-x_{m-2}|+\cdots+|x_{n+1}-x_n|\le|x_2-x_1|\cdot(\alpha^{m-2}+\alpha^{m-3}+\cdots+\alpha^{n-1})$

$=|x_2-x_1|\cdot\dfrac{\alpha^{n-1}(1-\alpha^{m-n})}{1-\alpha}<\dfrac{\alpha^{n-1}|x_2-x_1|}{1-\alpha}$

$\because 0<\alpha<1$ $\therefore \lim\limits_{n\to\infty}\alpha^n=0$. $\therefore \lim\limits_{n\to\infty}\dfrac{\alpha^{n-1}}{1-\alpha}|x_2-x_1|=0$

即$\forall\varepsilon>0$. $\exists N$, $n>N$时 $\dfrac{\alpha^{n-1}}{1-\alpha}|x_2-x_1|<\varepsilon$.

$\therefore m>n>N$时. $|x_m-x_n|<\varepsilon$. 由柯西收敛原理 $\{x_n\}$收敛.

12. $\begin{cases}x_{n+1}=\dfrac{x}{2}-\dfrac{x_n^2}{2}\\ x_n=\dfrac{x}{2}-\dfrac{x_{n-1}^2}{2}\end{cases}\Rightarrow |x_{n+1}-x_n|=\dfrac{|x_n^2-x_{n-1}^2|}{2}=\dfrac{|x_n+x_{n-1}|}{2}\cdot|x_n-x_{n-1}|$

$\because x_{n+1}=\dfrac{x}{2}-\dfrac{x_n^2}{2}\le\dfrac{x}{2}\le\dfrac{1}{2}$. $\therefore x-x_n^2\ge x-\dfrac{x^2}{4}=x(1-\dfrac{x}{4})>0$ $(0\le x\le1)$

$\therefore x_{n+1}=\dfrac{1}{2}(x-x_n^2)>0$. $\therefore 0<x_{n+1}\le\dfrac{1}{2}$ $\therefore |x_n+x_{n-1}|\le1$.

$\therefore |x_{n+1}-x_n|\le\dfrac{1}{2}|x_n-x_{n-1}|$. 由习题 2(A)-11 知$\{x_n\}$收敛. 设$\lim\limits_{n\to\infty}x_n=A$

在 $x_{n+1}=\dfrac{x}{2}-\dfrac{x_n^2}{2}$中 两侧取极限. $A=\dfrac{x}{2}-\dfrac{A^2}{2}\Rightarrow A=\sqrt{1+x}-1$

13. (1) $|x_{n+1}-x_n| = \left|\frac{1}{1+x_n}-\frac{1}{1+x_{n-1}}\right| = \frac{1}{|(1+x_n)(1+x_{n-1})|}\cdot|x_n-x_{n-1}|$

显然 $x_n>0$　$\therefore x_n=\frac{1}{1+x_{n-1}}<1$.　$\therefore x_{n+1}=\frac{1}{1+x_n}>\frac{1}{1+1}=\frac{1}{2}$.

$\therefore \frac{1}{|(1+x_n)(1+x_{n-1})|}<\frac{1}{\frac{3}{2}\cdot\frac{3}{2}}=\frac{4}{9}$　即 $|x_{n+1}-x_n|<\frac{4}{9}|x_n-x_{n-1}|$.

由习题2(A)-11. $\{x_n\}$收敛. 设 $\lim\limits_{n\to\infty}x_n=A$.

在 $x_{n+1}=\frac{1}{1+x_n}$ 两侧取极限, $A=\frac{1}{1+A}$.　$\therefore A^2+A=1$　$A=\frac{-1\pm\sqrt{5}}{2}$　$\because 0<A<1$　$\therefore A=\frac{\sqrt{5}-1}{2}$

(2) 记 $x_n=\frac{F_n}{F_{n+1}}$　$\therefore x_1=\frac{F_1}{F_2}=1$.　由 $F_{n+2}=F_n+F_{n+1} \Rightarrow \frac{F_{n+2}}{F_{n+1}}=\frac{F_n}{F_{n+1}}+1 \Rightarrow \frac{1}{x_{n+1}}=1+x_n$

$\therefore x_{n+1}=\frac{1}{1+x_n}$.　由(1)知 $\left\{\frac{F_n}{F_{n+1}}\right\}$ 收敛且极限为 $\frac{\sqrt{5}-1}{2}$.

14. $\boxed{\text{引理}}$ 对 $\forall\varepsilon>0$. 取 $k=\left[\frac{1}{\varepsilon}\right]$. 记 $S=\{1,2,\cdots,k\}$. 则 $\left\{\frac{p}{q}\,\middle|\,0<\left|\frac{p}{q}-x_0\right|<1, q\in S\right\}$ 是有限集. ($\Leftrightarrow$ $(x_0-1,0)\cup(0,x_0+1)$ 内. 满足 $\frac{1}{q}\geq\varepsilon$ 的点 $\frac{p}{q}$ 有有限个.)

proof: $\because x_0-1<\frac{p}{q}<x_0+1$　$\therefore q(x_0-1)<p<q(x_0+1)$　又 q 取值有限　$\therefore p$ 取值也有限　$\therefore \frac{p}{q}$ 取值为有限集. #

证明: 记 $\left\{\frac{p}{q}\,\middle|\,0<\left|\frac{p}{q}-x_0\right|<1, q\in S\right\}=\{x_1,x_2,\cdots,x_m\}$.　其中 $S=\left\{1,2,\cdots,\left[\frac{1}{\varepsilon}\right]\right\}$

$\therefore$ 在 x_0 的1半径去心邻域中. 只有有限的 m 个点 $x_1,\cdots,x_m$ 满足 $q\leq\left[\frac{1}{\varepsilon}\right]\leq\frac{1}{\varepsilon}$ 即 $\frac{1}{q}\geq\varepsilon$.

$\therefore$ 取 $\delta=\min\limits_{1\leq i\leq m}|x_i-x_0|$.　$\therefore$ 对 $\forall\varepsilon>0$. $\exists\delta$. 当 $0<|x-x_0|<\delta$ 时. 若 $x=\frac{p}{q}$ (x为有理数) 则 $\frac{1}{q}<\varepsilon$

对 $R(x)$ 而言. $\because R(x)=\begin{cases}\frac{1}{q} & x=\frac{p}{q}\\ 0 & x\text{为无理数}\end{cases}$　$\therefore$ 对 $\forall\varepsilon>0$. 取上述的 δ. 当 $0<|x-x_0|<\delta$ 时.

① 若 x 为无理数. $|R(x)|=0<\varepsilon$.

② 若 $x=\frac{p}{q}$　$|R(x)|=\frac{1}{q}<\varepsilon$.　$\therefore \lim\limits_{x\to x_0}R(x)=0$

15. $\Rightarrow$ $\because \lim\limits_{x\to x_0^+} f(x)=A$ $\therefore$ 对 $\forall \varepsilon>0$. $\exists \delta>0$. $x_0<x<x_0+\delta$ 时. 有 $|f(x)-A|<\varepsilon$.

$\therefore$ 对任意在 $(x_0, x_0+\delta)$ 中严减且以 x_0 为极限的数列 $\{x_n\}$ 中.

由于 $x_0<x<x_0+\delta$. $\lim\limits_{n\to\infty} x_n = x_0$ $\therefore$ 对 $\forall \varepsilon>0$. $\exists N$, $n>N$ 时 $|x_n-x_0|<\varepsilon$.

取 $\varepsilon=\delta$. $\therefore x_0-\delta<x_n<x_0+\delta$ $\therefore x_0<x_n<x_0+\delta$ $\therefore |f(x_n)-A|<\varepsilon$.

即对 $\forall \varepsilon>0$ $\exists N=N(\varepsilon)$. $n>N$ 时 $|f(x_n)-A|<\varepsilon$ 即 $\lim\limits_{n\to\infty} f(x_n)=A$.

$\Leftarrow$ 反设 $\lim\limits_{x\to x_0^+} f(x)\neq A$. 即 $\exists \varepsilon_0>0$. 对 $(0,\delta)$ 间的 $\forall \eta$, 均 $\exists$ $(x_0, x_0+\eta)$ 间的 x_η, 使 $|f(x_\eta)-A|\geq \varepsilon_0$

- 对 $\eta_1=\frac{\delta}{2}$. 取 x_1 满足 $x_0<x_1<x_0+\frac{\delta}{2}$. 进而 $|f(x_1)-A|\geq\varepsilon_0$.
- 对 $\eta_2=\min\{\frac{\delta}{4}, x_1-x_0\}$. 取 x_2 满足 $x_0<x_2<x_0+\frac{\delta}{4}$ 进而 $|f(x_2)-A|\geq\varepsilon_0$
- 对 $\eta_3=\min\{\frac{\delta}{8}, x_2-x_0\}$ 取 x_3 满足 $x_0<x_3<x_0+\frac{\delta}{8}$ 进而 $|f(x_3)-A|\geq\varepsilon_0$

……

得到数列 $\{x_n\}$ 满足 $x_0<x_n<x_0+\frac{\delta}{2^n}$ 且 $0<x_n-x_0<x_{n-1}-x_0$. 同时 $|f(x_n)-A|\geq\varepsilon_0$ 对 $\forall n$ 成立.

$\therefore \{x_n\}$ 是一个严格递减且极限为 x_0 的数列, 但 $f(x_n)$ 不收敛于 A. 矛盾. $\therefore \lim\limits_{x\to x_0^+} f(x)=A$.

16. 反设 $a^2+b^2\neq 0$.

$\therefore a\sin x+b\cos x=\sqrt{a^2+b^2}\sin(x+\varphi)$. $\because \lim\limits_{x\to+\infty}\sin x$ 不存在 $\therefore \lim\limits_{x\to+\infty}(a\sin x+b\cos x)$ 不存在.

矛盾. $\therefore a=b=0$.

17. 设 $f(x)$ 周期为 T 即 $f(x)=f(x+nT)$.

记 $x_n=x+nT$. $\{x_n\}$ 的极限为 $+\infty$.

$\because \lim\limits_{x\to+\infty} f(x)=0$. 由海涅定理 $\lim\limits_{n\to\infty} f(x_n)=\lim\limits_{n\to\infty} f(x+nT)=\lim\limits_{n\to\infty} f(x)=0$. $\therefore f(x)=0$.

南開大學 作 业 纸

系别________ 班级________ 姓名________ 第 页

18. 假设$\exists x_0$ 使 $f(x_0)\neq A$.

记$x_n=2^n x_0$. $\because f(x_0)=f(2x_0)=f(4x_0)=\cdots=\cdots\neq A$.

$\{x_n\}$以∞为极限，且对$\forall n$, $f(x_n)=f(2^n x_0)=f(x_0)\neq A$. $\therefore \lim\limits_{n\to\infty} f(x_n)\neq \lim\limits_{x\to+\infty} f(x)$.

与海涅定理矛盾 $\therefore f(x)\equiv A$.

19. 由题意 $\exists \delta>0$. $M>0$. $|x|<\delta$时. $|f(x)|\leq M$.

在$(-\delta,\delta)$上. $|f(x)|=\frac{|f(ax)|}{b}\leq\frac{M}{b}$

在$(-\frac{\delta}{a},\frac{\delta}{a})$上 $\because ax\in(-\delta,\delta)$ $\therefore |f(x)|=\frac{|f(ax)|}{b}\leq\frac{1}{b}\cdot\frac{M}{b}=\frac{M}{b^2}$

在$(-\frac{\delta}{a^2},\frac{\delta}{a^2})$上 $\because ax\in(-\frac{\delta}{a},\frac{\delta}{a})$ $\therefore |f(x)|=\frac{|f(ax)|}{b}\leq\frac{1}{b}\cdot\frac{M}{b^2}=\frac{M}{b^3}$

……

对$\forall n\in N$ 在$(-\frac{\delta}{a^n},\frac{\delta}{a^n})$上. $|f(x)|\leq\frac{M}{b^{n+1}}$

$\because \lim\limits_{n\to\infty}\frac{M}{b^{n+1}}=0$ $(b>1)$ $\therefore$ 对$\forall\varepsilon>0$. $\exists N$. $n>N$时 $\frac{M}{b^n}<\varepsilon$.

取$\delta_0=\frac{\delta}{a^N}$ $\therefore$ 当$0<|x|<\delta_0$时. $|f(x)|\leq\frac{M}{b^N}<\varepsilon$. $\therefore \lim\limits_{x\to0}f(x)=0$.

20. 对$\forall\varepsilon>0$. 取$\eta=f(a+\varepsilon)-f(a)$. 不妨设$f$递增 $\therefore \eta>0$.

$\because \lim\limits_{n\to\infty}f(x_n)=f(a)$. 由于$\eta>0$. $\therefore\exists N$. $n>N$时 $|f(x_n)-f(a)|<\eta$

$\because x_n\in(a,b)$ $\therefore$ $f(x_n)>f(a)$ $\therefore$ $0<f(x_n)-f(a)<f(a+\varepsilon)-f(a)$

$\therefore$ $f(a)<f(x_n)<f(a+\varepsilon)$ 由f单增 $\therefore a<x_n<a+\varepsilon$. $\therefore |x_n-a|<\varepsilon$

$\therefore \lim\limits_{n\to\infty}x_n=a$

南開大學 作业纸

系别______ 班级______ 姓名______ 第 页

21. 对 $\forall x_0\in(a,b)$ 设集合 $A=\{f(x)\mid x\in(a,x_0)\}$. 显然A有上界.

由确界原理(课本P17). A存在上确界 M. 即:

- 对一切 $x\in(a,x_0)$, $f(x)\le M$.
- 对 $\forall\varepsilon>0$. $\exists x_1\in(a,x_0)$, $f(x_1)>M-\varepsilon$.

取 $\delta=x_0-x_1>0$. ∴当 $-\delta<x-x_0<0$ 时. 即 $x_1<x<x_0$. ∴ $f(x_1)>M-\varepsilon$.

∴ $M-\varepsilon<f(x_1)<M<M+\varepsilon$. 即 $|f(x_1)-M|<\varepsilon$. ∴ $\lim\limits_{x\to x_0^-}f(x)=M$.

同理. $\lim\limits_{x\to x_0^+}f(x)=M'$ ∵ $f(x)$ 递增. ∴ $M\le f(x_0)\le M'$.

22. $n\to\infty$ 时

$$\frac{f(2^n x)}{f(x)}=\frac{f(2^n x)}{f(2^{n-1}x)}\cdot\frac{f(2^{n-1}x)}{f(2^{n-2}x)}\cdots\frac{f(2x)}{f(x)}\to1 \quad 即 \lim_{x\to+\infty}\frac{f(2^n x)}{f(x)}=1 \quad ∴ \lim_{x\to+\infty}\frac{f(\frac{x}{2^n})}{f(x)}=1$$

∵对 $\forall a>0$. 必存在 n 满足 $\frac{1}{2^n}<a<2^n$. 不妨设 $f(x)$ 单增. ∴ $f(\frac{x}{2^n})\le f(ax)\le f(2^n x)$

∴ $\frac{f(\frac{x}{2^n})}{f(x)}\le\frac{f(ax)}{f(x)}\le\frac{f(2^n x)}{f(x)}$ ∵ $\lim\limits_{x\to+\infty}\frac{f(2^n x)}{f(x)}=\lim\limits_{x\to+\infty}\frac{f(\frac{x}{2^n})}{f(x)}=1$. 由两边夹定理 $\lim\limits_{x\to+\infty}\frac{f(ax)}{f(x)}=1$

23. ① $\lim\limits_{n\to\infty}x_n=+\infty\Rightarrow\forall A>0$. $\exists N_1>0$. s.t. $\forall n\ge N_1$, $x_n>A$.

② $\lim\limits_{n\to\infty}f(x_n)=A\Rightarrow\forall\varepsilon>0$. $\exists N_2>0$. s.t. $\forall n\ge N_2$, $A-\varepsilon<f(x_n)<A+\varepsilon$.

对 $\forall\varepsilon>0$

⟶ 由②. $\exists N_2>0$. $f(x_{N_2})>A-\varepsilon$. ∵ f 单增 ∴对 $\forall x>x_{N_2}$. $f(x)>A-\varepsilon$.

⟶ 由① 对 $\forall x>x_{N_2}$. ∃正整数 $N_3>\max\{x,N_2\}$ s.t. $\forall n\ge N_3$ 有 $x_n>x$. 取 $n=N_3$, ∴ $x_{N_3}>x$

∴此时 $x_n\ge x_{N_3}>x>x_{N_2}$. 对 $\forall n\ge N_3>N_2$ 均成立.

由②, $f(x_n)<A+\varepsilon$. ∵ f 单增 ∴ $f(x)<f(x_n)<A+\varepsilon$.

综上: 对 $\forall\varepsilon>0$. $\exists N_2>0$. 对 $\forall x>x_{N_2}$, 有 $A-\varepsilon<f(x)<A+\varepsilon$. 即 $\lim\limits_{x\to+\infty}f(x)=A$.

24. ξ的存在性: $\because f: [a,b] \to [a,b]$. 取 $t \in [a,b]$.

$\therefore f(t) \in [a,b]$, $f(f(t)) \in [a,b]$, $\cdots$, $f_{(n)}(t) \in [a,b]$.

记 $x_1 = t$, $x_2 = f(t)$, $\cdots$, $x_n = f_{(n)}(t)$. 即 $f(x_n) = x_{n+1}$.

$\because \exists \alpha \in (0,1)$ 使 $|x_{n+2} - x_{n+1}| \le \alpha |x_{n+1} - x_n|$.

由习题 2(A)-11, $\{x_n\}$收敛. 设 $\lim\limits_{n\to\infty} x_n = A$. $A \in [a,b]$

$\because 0 \le |f(A) - f(x_n)| = |f(A) - x_{n+1}| \le \alpha |A - x_n|$.

两边求极限有 $0 \le |f(A) - A| \le \alpha |A - A| = 0$ $\therefore f(A) = A$. 即 f 存在不动点.

ξ的唯一性: 设 ξ、η 均是不动点且 $\xi \ne \eta$.

则 $0 \ne |\xi - \eta| = |f(\xi) - f(\eta)| \le \alpha |\xi - \eta|$ $\therefore \alpha \ge 1$. 与 $\alpha \in (0,1)$ 矛盾 $\therefore$ 不动点唯一

25. $\because \lim\limits_{n\to\infty} x_n = +\infty$ $\therefore$ 对 $\forall M > 0$. $\exists N_1$ 使 $n > N_1$ 时 $x_n > 2M+2$.

又 $\because \lim\limits_{n\to\infty} \dfrac{N}{n} = 0$, $\lim\limits_{n\to\infty} \dfrac{x_1 + \cdots + x_N}{n} = 0$. $\therefore \exists N_2$, $n > N_2$ 时 $\dfrac{N}{n} < \dfrac{1}{2}$ 且 $\dfrac{|x_1 + \cdots + x_N|}{n} < \dfrac{1}{2}$

取 $N = \max\{N_1, N_2\}$. $n > N$ 时.

$$\left|\frac{x_1 + \cdots + x_n}{n}\right| = \left|\frac{x_{N+1} + x_{N+2} + \cdots + x_n}{n} + \frac{x_1 + \cdots + x_N}{n}\right| > \left|\frac{|x_{N+1} + \cdots + x_n|}{n} - \frac{|x_1 + \cdots + x_N|}{n}\right|$$

$$> \left|\frac{n - N_1}{n}(2M+2) - \frac{1}{2}\right| > \left|M + 1 - \frac{1}{2}\right| = M + \frac{1}{2} > M$$

$$\therefore \lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} = +\infty.$$

26. ① $A=0$时. 由 $\lim\limits_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}=0$ 即 $\forall\varepsilon>0$. $\exists N$. 当$n>N$时 $\left|\frac{x_{n+1}-x_n}{y_{n+1}-y_n}\right|<\varepsilon$.

由于$\{y_n\}$递增 ∴ $y_{n+1}-y_n>0$ 即 $-\varepsilon(y_{n+1}-y_n)<x_{n+1}-x_n<\varepsilon(y_{n+1}-y_n)$

$$\therefore \left.\begin{array}{l} -\varepsilon(y_n-y_{n-1})<x_n-x_{n-1}<\varepsilon(y_n-y_{n-1}) \\ \quad\cdots\cdots \\ -\varepsilon(y_{N+2}-y_{N+1})<x_{N+2}-x_{N+1}<\varepsilon(y_{N+2}-y_{N+1}) \end{array}\right\} \text{相加}\Rightarrow \therefore -\varepsilon(y_n-y_{N+1})<x_n-x_{N+1}<\varepsilon(y_n-y_{N+1})$$

即 $\left|\frac{x_n-x_{N+1}}{y_n-y_{N+1}}\right|<\varepsilon$.

又∵ $\left|\frac{x_n-x_{N+1}}{y_n-y_{N+1}}\right|=\left|\frac{\frac{x_n}{y_n}-\frac{x_{N+1}}{y_n}}{1-\frac{y_{N+1}}{y_n}}\right|>\left|\frac{x_n}{y_n}-\frac{x_{N+1}}{y_n}\right|$

∴ $\left|\frac{x_n}{y_n}\right|=\left|\frac{x_n}{y_n}-\frac{x_{N+1}}{y_n}+\frac{x_{N+1}}{y_n}\right|<\left|\frac{x_n}{y_n}-\frac{x_{N+1}}{y_n}\right|+\left|\frac{x_{N+1}}{y_n}\right|<\left|\frac{x_n-x_{N+1}}{y_n-y_{N+1}}\right|+\left|\frac{x_{N+1}}{y_n}\right|<\varepsilon+\left|\frac{x_{N+1}}{y_n}\right|$

而∵ $y_n\to+\infty$. ∴ $\lim\limits_{n\to\infty}\left|\frac{x_{N+1}}{y_n}\right|=0$. 即 $\exists N_1>N$. $n>N_1$时 $\left|\frac{x_{N+1}}{y_n}\right|<\varepsilon$.

∴对$\forall\varepsilon>0$. $\exists N_1$. $n>N_1$时. $\left|\frac{x_n}{y_n}\right|<\varepsilon+\varepsilon=2\varepsilon$ ∴ $\lim\limits_{n\to\infty}\frac{x_n}{y_n}=0$ 得证.

② $A\in\mathbb{R}\backslash\{0\}$时.

注意到 $\lim\limits_{n\to\infty}\left(\frac{x_{n+1}-x_n}{y_{n+1}-y_n}-A\right)=\lim\limits_{n\to\infty}\frac{x_{n+1}-Ay_{n+1}-(x_n-Ay_n)}{y_{n+1}-y_n}=0$

令 $z_n=x_n-Ay_n$. 由①. 可知 $\lim\limits_{n\to\infty}\frac{z_n}{y_n}=0$ ∴ $\lim\limits_{n\to\infty}\frac{x_n}{y_n}=A$.

③ $A=+\infty$时.

$\lim\limits_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}=+\infty>1$. 由极限保序性. $\exists N$. 当$n>N$时 $\frac{x_{n+1}-x_n}{y_{n+1}-y_n}>1$.

∴ $x_{n+1}-x_n>y_{n+1}-y_n>0$ 模仿①中步骤可得 $x_{n+1}-x_{N+1}>y_{n+1}-y_{N+1}$ ∴$\{x_n\}$是递增的无穷大量

由 $\lim\limits_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}=+\infty$ 知 $\lim\limits_{n\to\infty}\frac{y_{n+1}-y_n}{x_{n+1}-x_n}=0$. 又$\{x_n\}$是递增无穷大量. 由① $\lim\limits_{n\to\infty}\frac{y_n}{x_n}=0$ ∴ $\lim\limits_{n\to\infty}\frac{x_n}{y_n}=+\infty$

④ $A=-\infty$时. 取 $z_n=-x_n$. 由③可得 $\lim\limits_{n\to\infty}\frac{x_n}{y_n}=-\infty$.

南開大學 作业纸

系别________ 班级________ 姓名________ 第 页

27. $x_{n+1}-x_n=-x_n^2\le 0$ $\therefore\{x_n\}$递减. 又由归纳法易知 $0<x_n<1$.

$\therefore$由单调收敛定理 $\{x_n\}$收敛. 设 $\lim\limits_{n\to\infty}x_n=A$. 对 $x_{n+1}=x_n(1-x_n)$两侧取极限: $A=A(1-A)$ $\therefore A=0$.

即 $\lim\limits_{n\to\infty}x_n=0$. $\therefore\lim\limits_{n\to\infty}\frac{1}{x_n}=+\infty$ 即$\{\frac{1}{x_n}\}$是递增的无穷大量.

又: $x_{n+1}=x_n(1-x_n)\Rightarrow\frac{1}{x_{n+1}}=\frac{1}{x_n}+\frac{1}{1-x_n}\Rightarrow\frac{1}{1-x_n}=\frac{1}{x_{n+1}}-\frac{1}{x_n}$

$\therefore\lim\limits_{n\to\infty}\left(\frac{1}{x_{n+1}}-\frac{1}{x_n}\right)=\lim\limits_{n\to\infty}\frac{1}{1-x_n}=1$ $\therefore\lim\limits_{n\to\infty}\frac{(n+1)-n}{\frac{1}{x_{n+1}}-\frac{1}{x_n}}=1$.

又$\because\{\frac{1}{x_n}\}$是递增的无穷大量. 由Stolz定理 $\lim\limits_{n\to\infty}\frac{n}{\frac{1}{x_n}}=\lim\limits_{n\to\infty}nx_n=1$.