第12章 重积分

练习12.1

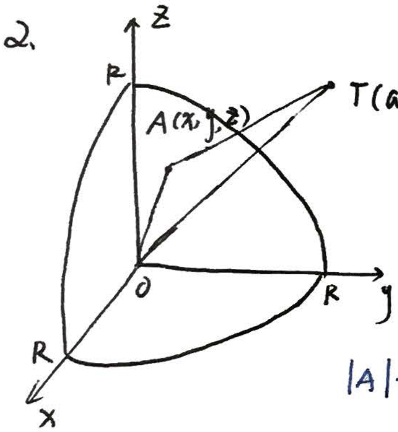
1. 证：由积分中值定理，$\exists(\xi_i,\eta_i)\in D$

$$\text{s.t. } \iint_D f(x,y)\,dxdy = f(\xi_i,\eta_i)\cdot V_J(D) = \pi r^2\cdot f(\xi_i,\eta_i)$$

$\because (\xi_i,\eta_i)\in D \quad \therefore r\to 0$ 时 $f(\xi_i,\eta_i)\to f(x_0,y_0)$ （$f(x)$ 在D上连续）

$$\therefore \lim_{r\to 0}\frac{1}{\pi r^2}\iint_D f(x,y)\,dxdy = f(x_0,y_0)$$

2.




第一封限内示意图见左，其中A点在球$x^2+y^2+z^2=R^2$内，$|OT|=|A|$.

在$\triangle OTA$上. $\because OT-OA\le AT\le OT+OA$

$$|A|-R\le |A|-\sqrt{x^2+y^2+z^2}\le\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}\le |A|+\sqrt{x^2+y^2+z^2}\le |A|+R$$

$$\therefore \frac{1}{|A|+R}\le\frac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}\le\frac{1}{|A|-R}$$

$$\therefore \quad I=\iiint_V\frac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}\le\iiint_V\frac{dxdydz}{|A|-R}=\frac{1}{|A|-R}\cdot V_J(V)=\frac{4\pi}{3}\cdot\frac{R^3}{|A|-R}$$

同理 $I\ge\frac{4}{3}\pi R^3\cdot\frac{1}{|A|+R}$

$$\therefore \frac{4\pi}{3}\cdot\frac{R^3}{|A|+R}\le I\le\frac{4\pi}{3}\cdot\frac{R^3}{|A|-R}$$

3. 证：由于 $f(x)\geqslant 0$ 且不恒为零，$\therefore \exists x_0\in D$ s.t. $f(x_0)>0$

$\because f(x)$ 在 C 上连续 $\therefore \exists \delta>0$ 当 $x\in B_\delta(x_0)\cap D$ 时，$f(x)>\frac{f(x_0)}{2}$

记 $H=B_\delta(x_0)\cap D$，显然 H 是有界闭集且可测

$$\therefore \int_H f(x)d\Omega = f(\xi)\cdot V_J(H)\geqslant \frac{f(x_0)}{2}\cdot V_J(H)>0$$

$$\therefore \int_D f(x)d\Omega \geqslant \int_H f(x)d\Omega>0$$

4. 证：不妨设 $g(x)$ 在 D 上恒非负。$\because f\in C(D)$ $\therefore f$ 在 D 上取得最值，分别设为 M 与 m

$$\therefore mg(x)\leqslant f(x)g(x)\leqslant Mg(x)$$

$$\therefore m\int_D g(x)d\Omega \leqslant \int_D f(x)g(x)d\Omega \leqslant M\int_D g(x)d\Omega$$

当 $g(x)\neq 0$ 时，有 $m\leqslant \dfrac{\int_D f(x)g(x)d\Omega}{\int_D g(x)d\Omega}\leqslant M$.

记 $h(x)=\dfrac{\int_D f(x)g(x)d\Omega}{\int_D g(x)d\Omega}$ $\therefore h(x)=\lambda$ 在 D 上有解. $(\lambda\in[m,M])$

$\because f\in C(D)$ $\therefore \exists \xi\in D$ s.t. $f(\xi)=\lambda$

$$\therefore f(\xi)=\lambda=h(x)$$

$$\therefore \int_D f(x)g(x)d\Omega = f(\xi)\int_D g(x)d\Omega.$$

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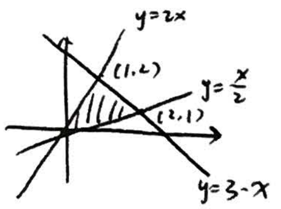
练习 12.2

1. $\int \frac{x^2-y^2}{(x^2+y^2)^2}dy = \frac{y}{x^2+y^2}+C$　　$\therefore \int_0^1 dx\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dy = \int_0^1 \frac{1}{1+x^2}dx = \arctan 1 = \frac{\pi}{4}$

$\int \frac{x^2-y^2}{(x^2+y^2)^2}dx = \frac{-x}{x^2+y^2}+C$　　$\therefore \int_0^1 dy\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dx = \int_0^1 \frac{-1}{1+y^2}dy = -\arctan 1 = -\frac{\pi}{4}$

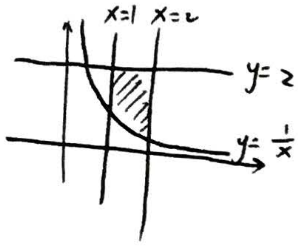
等式不成立.

2. (1)



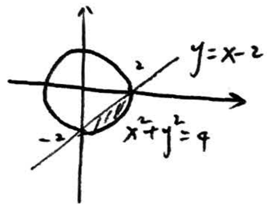

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 dx\int_{\frac{x}{2}}^{2x} f(x,y)dy + \int_1^2 dx\int_{\frac{x}{2}}^{3-x} f(x,y)dy$$
$$= \int_0^1 dy\int_{\frac{y}{2}}^{2y} f(x,y)\,dx + \int_1^2 dy\int_{\frac{y}{2}}^{3-y} f(x,y)\,dx$$

(2)



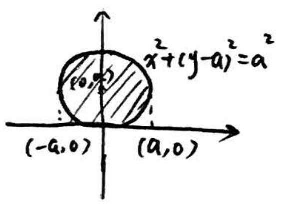

$$\iint_D f(x,y)\,dx\,dy = \int_1^2 dx\int_{\frac{1}{x}}^{2} f(x,y)\,dy$$
$$= \int_{\frac{1}{2}}^1 dy\int_{\frac{1}{y}}^{2} f(x,y)\,dx + \int_1^2 dy\int_1^2 f(x,y)\,dx$$

(3)



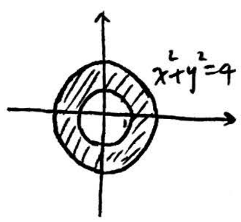

$$\iint_D f(x,y)\,dx\,dy = \int_0^2 dx\int_{-\sqrt{4-x^2}}^{x-2} f(x,y)\,dy = \int_{-2}^0 dy\int_{y+2}^{\sqrt{4-y^2}} f(x,y)\,dx$$

(4)




$$\iint_D f(x,y)\,dx\,dy = \int_{-a}^{a} dx\int_{a-\sqrt{a^2-x^2}}^{a+\sqrt{a^2-x^2}} f(x,y)\,dy = \int_0^{2a} dy\int_{-\sqrt{2ay-y^2}}^{\sqrt{2ay-y^2}} f(x,y)\,dx$$

(5)




$$\iint_D f(x,y)\,dx\,dy = \int_{-2}^{-1} dx\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)dy + \int_{-1}^{1} dx\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} f(x,y)dy$$
$$+ \int_{-1}^{1} dx\int_{-\sqrt{4-x^2}}^{-\sqrt{1-x^2}} f(x,y)dy + \int_1^2 dx\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)dy$$
$$= \int_{-2}^{-1} dy\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x,y)dx + \int_{-1}^{1} dy\int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} f(x,y)dx$$
$$+ \int_{-1}^{1} dy\int_{-\sqrt{4-y^2}}^{-\sqrt{1-y^2}} f(x,y)dx + \int_1^2 dy\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x,y)dx$$

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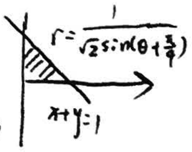
3.

(1) 原式 $= \int_1^2 dx \int_{\frac{1}{x}}^{2} (x^2-3xy)\,dy = \int_1^2 dx \left(x^2y - \frac{3}{2}xy^2\right)\Big|_{\frac{1}{x}}^{2} = \int_1^2 \left(2x^2-7x+\frac{3}{2x}\right)dx$

$= \frac{2}{3}x^3 - \frac{7}{2}x^2 + \frac{3}{2}\ln x \Big|_1^2 = \frac{3}{2}\ln 2 - \frac{35}{6}$

(2)

(a) 


原式 $= \iint_{D'} r^3\,dr\,d\theta = \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{1}{\sqrt{2}\sin(\theta+\frac{\pi}{4})}} r^3\,dr$

$= \frac{1}{16}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin^4(\theta+\frac{\pi}{4})}$

$\because \int \frac{dx}{\sin^4(x+\frac{\pi}{4})} = \int \frac{dy}{\sin^4 y} = \int \frac{\sin^2 y + \cos^2 y}{\sin^4 y}\,dy$

$= \int \csc^2 x\,dx + \int \cot^2 x \csc^2 x\,dx$

$= -\cot x - \int \cot^2 x\,d\cot x = -\cot x - \frac{1}{3}\cot^3 x$

$\therefore$ 原式 $= \frac{-1}{16}\cdot\left(\cot x + \frac{1}{3}\cot^3 x\right)\Big|_{\frac{\pi}{4}}^{\frac{3}{4}\pi} = \frac{1}{6}$

(b) 设 $\begin{cases} x = a\cdot r\cos\theta \\ y = b\cdot r\sin\theta \end{cases}$ $0\le r\le 1,\ 0\le\theta\le 2\pi$

由对称性

$$\iint_D (x^2+y^2)\,dx\,dy = 4\int_0^1 abr\cdot dr \int_0^{\frac{\pi}{2}} r^2(a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta$$

$$= \frac{\pi}{4}ab(a^2+b^2)$$

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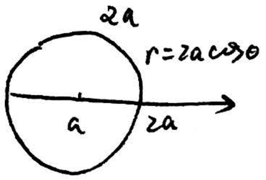
4、

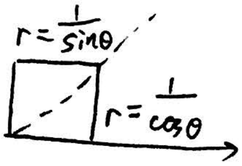
(1)




作 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$\iint_D f(x,y)\,dxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} f(r\cos\theta, r\sin\theta)\, r\,dr$$

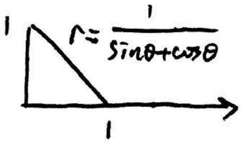
(2)




作 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$\iint_D f(x,y)\,dxdy = \int_0^{\frac{\pi}{4}} d\theta \int_0^{\frac{1}{\cos\theta}} f(r\cos\theta, r\sin\theta)\, r\,dr + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^{\frac{1}{\sin\theta}} f(r\cos\theta, r\sin\theta)\, r\,dr$$

(3)




作 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$\iint_D f(x,y)\,dxdy = \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{1}{\sin\theta+\cos\theta}} f(r\cos\theta, r\sin\theta)\,dr$$

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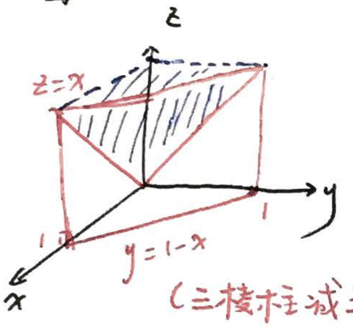
练习 12.3

1. 积分域如图:

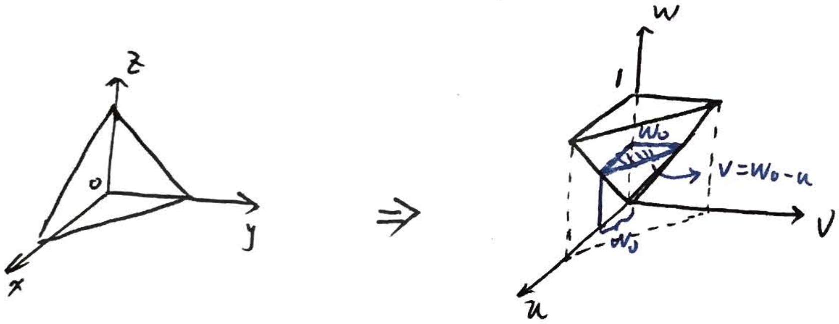



(三棱柱减三棱锥)

$$\int_0^1 dx\int_0^{1-x} dy\int_0^{x+y} f(x,y,z)\,dz$$

$$=\int_0^1 dy\int_0^1 dz\int_0^{1-y} f\,dx-\int_0^1 dy\int_y^1 dz\int_0^{z-y} f\,dx$$

2. (1)




令 $u=x,\ v=y,\ w=x+y+z,\ \dfrac{D(x,y,z)}{D(u,v,w)}=\begin{vmatrix}1&0&0\\0&1&0\\-1&-1&1\end{vmatrix}=1$

$$\therefore \iiint_V \frac{dx\,dy\,dz}{(x+y+z+1)^3}=\iiint_{V'}\frac{du\,dv\,dw}{(w+1)^3}=\int_0^1 dw\iint_{D(w)}\frac{1}{(w+1)^3}\,du\,dv$$

$$=\int_0^1\frac{1}{(w+1)^3}dw\int_0^w du\int_0^{w-u}dv=\int_0^1\frac{1}{(w+1)^3}dw\int_0^w (w-u)\,du$$

$$=\int_0^1\frac{1}{(w+1)^3}\cdot\left(-\frac{(w-u)^2}{2}\Big|_0^w\right)dw=\int_0^1\frac{w^2}{2(w+1)^3}dw$$

$$=\frac12\int_0^1\left(\frac{1}{w+1}-\frac{2}{(w+1)^2}+\frac{1}{(w+1)^3}\right)dw=\frac12\cdot\left(\ln(w+1)+\frac{2}{w+1}-\frac{1}{2(w+1)^2}\right)\Big|_0^1=\frac12\left(\ln 2-\frac58\right)$$

(2)

$V':\ r\le 1$

令 $x=ar\sin\varphi\cos\theta$

$y=br\sin\varphi\sin\theta$

$z=cr\cos\varphi$

则 $\iiint_V\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c}}\,dx\,dy\,dz=\iiint_{V'}\sqrt{1-r^2}\,abc\,r^2\sin\varphi\,dr\,d\theta\,d\varphi$

$$=abc\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\,d\varphi\int_0^1\sqrt{1-r^2}\,r^2\,dr=\frac{\pi^2}{4}abc$$

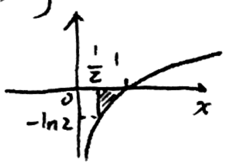
习题12

1. 记 $D=\{(t,u)\mid 0\le u\le x,\ 0\le t\le u\}=\{(t,u)\mid t\le u\le x,\ 0\le t\le x\}$

记 $F(t,u)=f(t)$

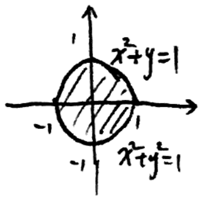
$$\therefore \int_0^x du\int_0^u f(t)\,dt=\iint_D F(t,u)\,dt\,du=\int_0^x dt\int_t^x f(t)\,du$$
$$=\int_0^x (x-t)f(t)\,dt$$

2. (1)



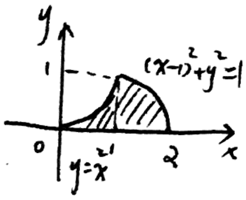

$$\int_1^{\frac{1}{2}} dx\int_0^{\ln x} f(x,y)\,dy=\int_{-\ln 2}^{0} dy\int_{\frac{1}{2}}^{e^y} f(x,y)\,dx.$$

(2)



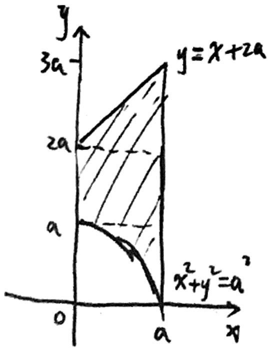

$$\int_{-1}^1 dx\int_{-\sqrt{1-x^2}}^{1-x^2} f(x,y)\,dy=\int_{-1}^0 dy\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx+\int_0^1 dy\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)\,dx$$

(3)




$$\int_0^1 dx\int_0^{x^2} f(x,y)\,dy+\int_1^2 dx\int_0^{\sqrt{1-(x-1)^2}} f(x,y)\,dy=\int_0^1 dy\int_{\sqrt{y}}^{1+\sqrt{1-y^2}} f(x,y)\,dx$$

(4)




$$\int_0^a dx\int_{\sqrt{a^2-x^2}}^{x+2a} f(x,y)\,dy=\int_0^a dy\int_{\sqrt{a^2-y^2}}^{a} f(x,y)\,dx+\int_a^{2a} dy\int_0^a f(x,y)\,dx$$
$$+\int_{2a}^{3a} dy\int_{y-2a}^{a} f(x,y)\,dx.$$

3.

记 $\Delta = \int_a^b p(x)dx \cdot \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \cdot \int_a^b p(x)g(x)dx$

记 $D=[a,b]\times[a,b]$

① $\Delta = \int_a^b p(x)dx \int_a^b p(y)f(y)g(y)dy - \int_a^b p(x)f(x)dx \cdot \int_a^b p(y)g(y)dy$

$= \iint_D p(x)p(y)f(y)g(y)\,dxdy - \iint_D p(x)f(x)p(y)g(y)\,dxdy$

$= \iint_D p(x)p(y)g(y)(f(y)-f(x))\,dxdy$

② $\Delta = \int_a^b p(y)dy \int_a^b p(x)f(x)g(x)\,dx - \int_a^b p(y)f(y)dy \int_a^b p(x)g(x)dx$

$= \iint_D p(y)p(x)f(x)g(x)\,dxdy - \iint_D p(y)f(y)p(x)g(x)\,dxdy$

$= \iint_D p(y)p(x)g(x)(f(x)-f(y))\,dxdy$

$\therefore \Delta = \frac{1}{2}\iint_D p(x)p(y)(g(y)-g(x))(f(y)-f(x))\,dxdy$

$\because$ $f(x)$、$g(x)$ 在 $[a,b]$ 上连续且单增 $\therefore (g(y)-g(x))(f(y)-f(x)) \geq 0$

$\because$ $p(x)$ 在 $[a,b]$ 上非负连续 $\therefore p(x)p(y) \geq 0$

$\therefore p(x)p(y)(g(y)-g(x))(f(y)-f(x)) \geq 0$

$\therefore \Delta \geq 0$.

习题 Q

4. 证：记 $p(x)=f^2(x)$，$g(x)=x$，$h(x)=\frac{1}{f(x)}$

∵ $f(x)$ 在 $[a,b]$ 上连续、单减、恒取正值

∴ $p(x)$ 在 $[a,b]$ 上非负连续，$g(x)$、$h(x)$ 在 $[a,b]$ 上连续、单增.

由 (3) 结论

$$\left(\int_a^b p(x)g(x)\,dx\right)\left(\int_a^b p(x)h(x)\,dx\right) \le \left(\int_a^b p(x)\,dx\right)\left(\int_a^b p(x)g(x)h(x)\,dx\right)$$

即 $$\int_a^b xf^2(x)dx \int_a^b f(x)dx \le \int_a^b f^2(x)dx \cdot \int_a^b xf(x)\,dx$$

∵ $p$、$g$、$h$ 均非负

∴ $$\frac{\int_a^b xf^2(x)dx}{\int_a^b xf(x)dx} \le \frac{\int_a^b f^2(x)dx}{\int_a^b f(x)dx}$$

5、 设D在x轴、y轴上的投影分别为 $[x_1,x_2]$ 与 $[y_1,y_2]$， $u=x_2-x_1$ $v=y_2-y_1$

由积分中值定理， $\exists (x_0,y_0)\in D$

使 $\iint_D (x-a)(y-b)\,dxdy = (x_0-a)(y_0-b)|D|$.

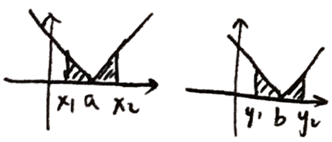
$\because (x_0,y_0)\in D$, $(a,b)\in D$ $\therefore |x_0-a|\le u$, $|y_0-b|\le v$

$\therefore \left|\iint_D (x-a)(y-b)\,dxdy\right| \le uv|D|$

又 $\left|\iint_D (x-a)(y-b)\,dxdy\right| \le \iint_D |x-a||y-b|\,dxdy$

$\le \iint_H |x-a||y-b|\,dxdy$ $\quad (H: [x_1,x_2]\times[y_1,y_2])$

$= \int_{x_1}^{x_2}|x-a|\,dx \cdot \int_{y_1}^{y_2}|y-b|\,dy$




$= \left[\frac{1}{2}(x_1-a)^2+\frac{1}{2}(x_2-a)^2\right]\left[\frac{1}{2}(y_1-b)^2+\frac{1}{2}(y_2-b)^2\right]$

$= \frac{1}{4}\left[(a-x_1)^2+(x_2-a)^2\right]\left[(b-y_1)^2+(y_2-b)^2\right]$

$\le \frac{1}{4}(x_2-x_1)^2(y_2-y_1)^2$

$= \frac{1}{4}u^2v^2$

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6. 证: 左式 $=\int_a^b f(x)\,dx\cdot\int_a^b \frac{dy}{f(y)} = \iint_H \frac{f(x)}{f(y)}\,dxdy$ （ $H: [a,b]\times[a,b]$ ）

$=\iint_H \frac{f(y)}{f(x)}\,dxdy$

∴左式 $=\frac{1}{2}\iint_H\left(\frac{f(x)}{f(y)}+\frac{f(y)}{f(x)}\right)dxdy \ge \frac{1}{2}\iint_H 2\,dxdy=\iint_H dxdy=(b-a)^2$

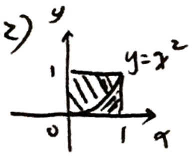
7.

(1)


$\iint_D e^{-y^2}dxdy=\int_0^1dy\int_0^y e^{-y^2}dx=\int_0^1 dy\cdot\left(x\cdot e^{-y^2}\right)\Big|_0^y=\int_0^1\left(y\cdot e^{-y^2}\right)dy$

$=\frac{1}{2}\int_0^1 e^{-y^2}dy^2=-\frac{1}{2}\int_0^1 e^{-y^2}d(-y^2)=-\frac{1}{2}\cdot e^x\Big|_0^{-1}=\frac{1}{2}-\frac{1}{2e}$

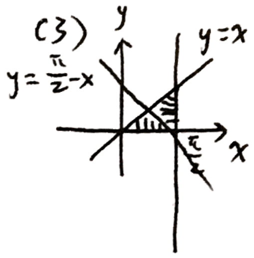
(2)




$\iint_D|y-x^2|\,dxdy=\int_0^1dx\int_{x^2}^1(y-x^2)dy+\int_0^1dx\int_0^{x^2}(x^2-y)dy$

$=\int_0^1\left(\frac{1}{2}x^4-x^2+\frac{1}{2}\right)dx+\int_0^1\left(\frac{1}{2}x^4\right)dx=\frac{11}{30}$

(3)




$\iint_D|\cos(x+y)|\,dxdy=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}dx\int_{\frac{\pi}{2}-x}^{x}-\cos(x+y)dy+\int_0^{\frac{\pi}{4}}dy\int_y^{\frac{\pi}{2}-y}\cos(x+y)dx$

$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1-\sin2x)dx+\int_0^{\frac{\pi}{4}}(1-\sin2y)dy=\int_0^{\frac{\pi}{2}}(1-\sin2x)dx=x+\frac{1}{2}\cos2x\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{2}-1$

(4)


$\int_0^1dy\int_y^1\frac{y}{\sqrt{1+x^3}}dx=\int_0^1 x\int_0^x\frac{y}{\sqrt{1+x^3}}dy=\int_0^1 x\cdot\left(\frac{y^2}{2\sqrt{1+x^3}}\right)\Big|_0^x$

$=\int_0^1\frac{x^2}{2\sqrt{1+x^3}}dx=\frac{1}{6}\int\frac{d(1+x^3)}{\sqrt{1+x^3}}=\frac{1}{6}\cdot2\sqrt{x}\Big|_1^2=\frac{\sqrt{2}-1}{3}$

# 南開大學 作业纸

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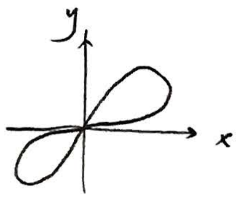
8.

(1)




作 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$ $\because (x^2+y^2)^2=2a^2xy \Rightarrow r^2=a^2\sin 2\theta$

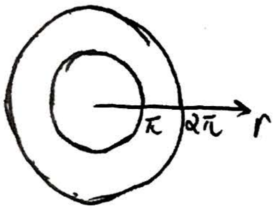
$$\therefore \iint_D (x^2-y^2)\,dx\,dy = \int_0^{\frac{\pi}{2}} d\theta \int_0^{a\sqrt{\sin 2\theta}} r^3\cos 2\theta\, dr$$

$$= \frac{a^4}{4}\int_0^{\frac{\pi}{2}} \sin^2 2\theta\cos 2\theta\, d\theta = \frac{a^4}{8}\int_0^{\frac{\pi}{2}} \sin^2 2\theta\, d\sin 2\theta$$

$$= \frac{a^4}{8}\cdot \left.\frac{\sin^3 2\theta}{3}\right|_0^{\frac{\pi}{2}} = 0$$

(由对称性亦可知)

(2)



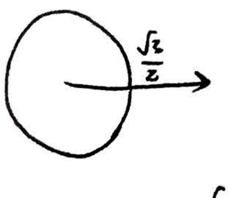

作 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$ $\pi^2 \le x^2+y^2 \le 4\pi^2 \Rightarrow \pi \le r \le 2\pi$

$$\therefore \iint_D e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi} d\theta \int_\pi^{2\pi} e^{-r^2} r\, dr$$

$$= \frac{1}{2}\int_0^{2\pi} d\theta \int_\pi^{2\pi} e^{-r^2} dr^2 = -\frac{1}{2}\int_0^{2\pi} (e^{-4\pi^2}-e^{-\pi^2})\, d\theta$$

$$= \pi(e^{-\pi^2}-e^{-4\pi^2})$$

(3)




作 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$ 令 $x^2+y^2 \le \sqrt{\frac{3}{4}-x^2-y^2}$ 得 $-\frac{\sqrt{2}}{2} \le r \le \frac{\sqrt{2}}{2}$

$$\therefore \iint_D 原式\, dx\,dy = \int_0^{\frac{\sqrt{2}}{2}} r^3\, dr \int_0^{2\pi} d\theta + \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{\frac{3}{4}-r^2}\, r\, dr \int_0^{2\pi} d\theta$$

$$= 2\pi\cdot\frac{1}{4}\cdot\frac{1}{4} - 2\pi\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\left.\left(\frac{3}{4}-r^2\right)^{\frac{3}{2}}\right|_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \frac{\pi}{8} + \frac{\pi}{12} = \frac{5}{24}\pi$$

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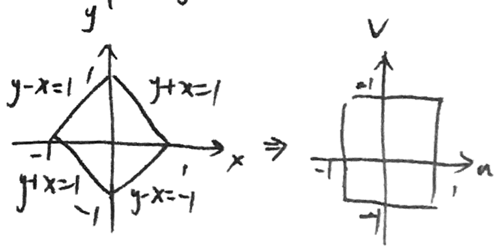
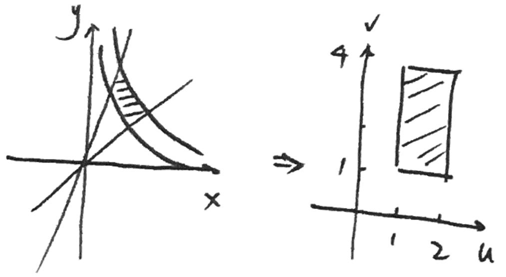
9、记 $\begin{cases} u=y+x \in [-1,1] \\ v=y-x \in [-1,1] \end{cases} \Rightarrow \begin{cases} x=\dfrac{u-v}{2} \\ y=\dfrac{u+v}{2} \end{cases} \quad \left|\dfrac{D(x,y)}{D(u,v)}\right| = \left|\begin{matrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{matrix}\right| = \dfrac{1}{2}$




$$\therefore \iint_D f(x+y)\,dx\,dy = \iint_{D'} f(u)\cdot\frac{1}{2}\,du\,dv$$

$$= \frac{1}{2}\int_{-1}^{1} dv \int_{-1}^{1} f(u)\,du = \int_{-1}^{1} f(u)\,du$$

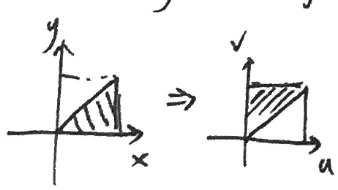
10. 记 $\begin{cases} u=xy \in [1,2] \\ v=\dfrac{y}{x} \in [1,4] \end{cases} \Rightarrow \begin{cases} x=\sqrt{\dfrac{u}{v}} \\ y=\sqrt{uv} \end{cases} \quad \left|\dfrac{D(x,y)}{D(u,v)}\right| = \left|\begin{matrix} \dfrac{1}{2\sqrt{uv}} & -\dfrac{\sqrt{u}}{2v\sqrt{v}} \\ \dfrac{\sqrt{v}}{2\sqrt{u}} & \dfrac{\sqrt{u}}{2\sqrt{v}} \end{matrix}\right| = \dfrac{1}{2v}$



$$\therefore \iint_D f(xy)\,dx\,dy = \iint_D f(u)\cdot\frac{1}{2v}\,du\,dv$$

$$= \int_1^4 \frac{1}{2v}\,dv \int_1^2 f(u)\,du = \ln 2\cdot\int_1^2 f(u)\,du$$

11、记 $\begin{cases} u=1-x \\ v=1-y \end{cases} \Rightarrow \begin{cases} x=1-u \\ y=1-v \end{cases}$ 且 $\left|\dfrac{D(x,y)}{D(u,v)}\right| = 1$ 同时 $D=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le x\}$ 代入 $u,v$，得 $D'=\{(u,v)\mid 0\le u\le 1,\ u\le v\le 1\}$



$$右式 = \iint_D f(1-x,1-y)\,dx\,dy = \iint_{D'} f(u,v)\,du\,dv$$

$$= \int_0^1 dv\int_0^v f(u,v)\,du$$

$$= \int_0^1 dx\int_0^x f(x,y)\,dy = 左式.$$

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12.

(1) 令 $\begin{cases} u = \frac{y^2}{x} \\ v = \frac{y}{x} \end{cases} \Rightarrow \begin{cases} x = \frac{u}{v^2} \\ y = \frac{u}{v} \end{cases}$ $\left|\frac{D(x,y)}{D(u,v)}\right| = \begin{vmatrix} \frac{1}{v^2} & \frac{-2u}{v^3} \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = \frac{u}{v^4}$

记原区域为D，则 $D'$: $[n,m]\times[1,2]$

$\therefore \iint_D 1\,dxdy = S = \iint_{D'} \frac{u}{v^4}\,dudv = \int_n^m u\,du \int_1^2 \frac{1}{v^4}\,dv = \frac{7}{48}(m^2-n^2)$

(2) 令 $\begin{cases} u = xy \\ v = xy^3 \end{cases} \Rightarrow \begin{cases} x = \sqrt{\frac{u^3}{v}} \\ y = \sqrt{\frac{v}{u}} \end{cases}$ $\left|\frac{D(x,y)}{D(u,v)}\right| = \begin{vmatrix} \frac{3\sqrt{u}}{2\sqrt{v}} & -\frac{1}{2}\sqrt{\frac{u^3}{v^3}} \\ -\frac{1}{2}\sqrt{\frac{v}{u^3}} & \frac{1}{2\sqrt{uv}} \end{vmatrix} = \frac{1}{2v}$

记原区域为D 则 $D'$: $[4,8]\times[5,15]$

$\therefore \iint_D 1\,dxdy = \iint_{D'} \frac{1}{2v}\,dudv = \int_4^8 du \int_5^{15} \frac{1}{2v}\,dv = 2\ln 3$

由对称性 $S = 4\ln 3$

(3) 令 $\begin{cases} x = ar\cos\theta \\ y = br\sin\theta \end{cases}$ $\therefore \left|\frac{D(x,y)}{D(r,\theta)}\right| = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr$

原曲线可化为 $r^2 = \frac{ab}{c^2}\sin\theta\cos\theta$ $\therefore$ 原区域$\frac{D}{2} \Rightarrow [0,\frac{\pi}{2}]\times[0,\frac{\sqrt{ab}}{c}\sqrt{\sin\theta\cos\theta}]$

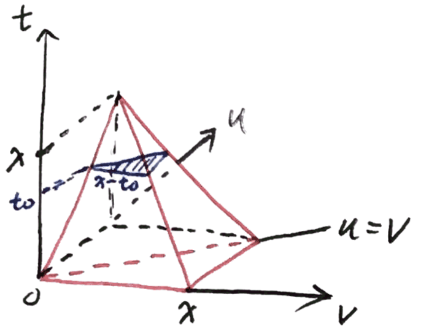
$\therefore \iint_D 1\,dxdy = \iint_{D'} abr\,drd\theta = \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\sqrt{ab\sin\theta\cos\theta}}{c}} abr\,dr = \frac{a^2b^2}{4c^2}$ 由对称性 $S = \frac{a^2b^2}{2c^2}$

(4) 令 $\begin{cases} u = a_1x+b_1y+c_1 \\ v = a_2x+b_2y+c_2 \end{cases}$ $\left|\frac{D(x,y)}{D(u,v)}\right| = 1/\left|\frac{D(u,v)}{D(x,y)}\right| = \frac{1}{|a_1b_2-a_2b_1|}$

区域D $\Rightarrow$ $D'$: $\{(u,v)\mid u^2+v^2\le h^2\}$

$\iint_{D'} \frac{1}{|a_1b_2-a_2b_1|}\,dudv = \frac{\pi h^2}{|a_1b_2-a_2b_1|} = S$

13、证明：




图中红色区域为 V.

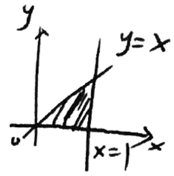
$$\int_0^x dv\int_0^v du\int_0^u f(t)dt = \iiint_V f(t)\,dv\,du\,dt$$

$$= \int_0^x dt\iint_{D'(t)} f(t)\,dv\,du = \int_0^x f(t)dt\int_0^{x-t}dv\int_0^v du$$

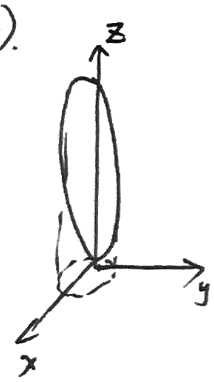
$$= \int_0^x f(t)dt\int_0^{x-t} v\,dv = \frac{1}{2}\int_0^x (x-t)^2 f(t)dt$$

14.

(1) $\iiint_V xy^2z^3\,dx\,dy\,dz = \iint_D xy^2\,dx\,dy\int_0^{xy} z^3\,dz$

D: 


$= \int_0^1 x\,dx\int_0^x y^2\,dy\int_0^{xy} z^3\,dz = \int_0^1 x\,dx\int_0^x \frac{1}{4}x^4y^6\,dy = \int_0^1 \frac{x^{12}}{28}\,dx = \frac{1}{364}$

(2). 


作柱坐标变换 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases}$

积分域在xoy投影为 $r=4\cos\theta$

积分域中 $\frac{r^2}{2} \le z \le 2r\cos\theta$

$\therefore \iiint_V (x^2+y^2)\,dx\,dy\,dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\int_0^{4\cos\theta} r^3\,dr\int_{\frac{r^2}{2}}^{2r\cos\theta} dz$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\int_0^{4\cos\theta} \left(2r^4\cos\theta - \frac{r^5}{2}\right) dr$

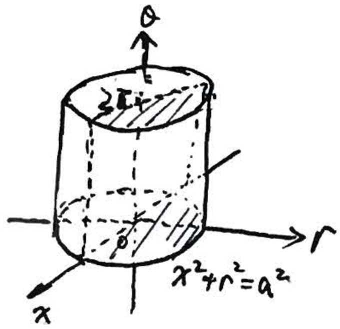
$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\cdot\left(\frac{2}{5}r^5\cos\theta - \frac{r^6}{16}\right)\Big|_0^{4\cos\theta}$

$= \frac{1024}{15}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6\theta\,d\theta = \frac{1024}{15}\cdot\left(\frac{5}{16}x + \frac{\sin 6x}{192} + \frac{3}{64}\sin 4x + \frac{15}{64}\sin 2x\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \frac{\cancel{1024}^{64}}{\cancel{15}_{3}}\cdot\frac{\cancel{5}^{1}}{\cancel{16}_{1}}\cdot\pi = \frac{64}{3}\pi$

(3) 令 $\begin{cases} x = x \\ y = r\cos\theta \\ z = r\sin\theta \end{cases}$

V':




$$\text{原式} = \iiint_{V'} \frac{r(b-x)}{((x-b)^2+r^2)^{\frac{3}{2}}}\,dx\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} \frac{r(b-x)}{((x-b)^2+r^2)^{\frac{3}{2}}}\,dr = \int_0^{2\pi} d\theta \int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} \frac{b-x}{2((x-b)^2+r^2)^{\frac{3}{2}}}\,d(x^2+r^2)$$

$$= \int_0^{2\pi} d\theta \int_{-a}^{a} dx \left(\frac{x-b}{\sqrt{(x-b)^2+r^2}}\right)\Bigg|_0^{\sqrt{a^2-x^2}}$$

$$= \int_0^{2\pi} d\theta \int_{-a}^{a} \left(\frac{x-a}{\sqrt{b^2+a^2-2bx}} - \frac{x-b}{|x-b|}\right) dx$$

对于 $\displaystyle\int_{-a}^{a} \frac{x-a}{\sqrt{b^2+a^2-2bx}}\,dx = \left(\frac{a^2}{2b} - \frac{b}{2}\right)\int \frac{dx}{\sqrt{b^2+a^2-2bx}} - \frac{1}{2b}\int \sqrt{b^2+a^2-2bx}\,dx$

$$= \left(\frac{1}{4} - \frac{a^2}{4b^2}\right)\int \frac{d(b^2+a^2-2bx)}{\sqrt{b^2+a^2-2bx}} + \frac{1}{4b^2}\int \sqrt{b^2+a^2-2bx}\,d(b^2+a^2-2bx)$$

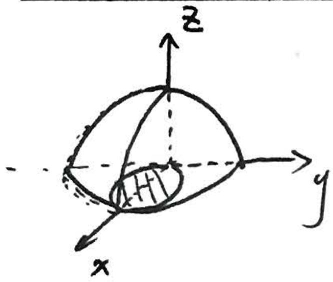
$$= \left(\frac{1}{4} - \frac{a^2}{4b^2}\right)\cdot 2\sqrt{b^2+a^2-2bx}\,\Big|_{-a}^{a} + \frac{1}{6b^2}\cdot(b^2+a^2-2bx)^{\frac{3}{2}}\Big|_{-a}^{a}$$

$$= \frac{(2b+a)(b-a)^2}{3b^2} - \frac{(2b-a)(b+a)^2}{3b^2} = \frac{2a^3}{3b^2} - 2a$$

而 $\displaystyle\int_{-a}^{a} \frac{x-b}{|x-b|}\,dx = 2a$

$$\therefore \text{原式} = \int_0^{2\pi} \frac{2a^3}{3b^2}\,d\theta = \frac{4\pi a^3}{3b^2}$$

(4)




作柱坐标变换 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases}$ $V'$: $\begin{cases} r^2+z^2\le a^2 \\ r\le a\cos\theta \\ z\geqslant 0 \end{cases}$

$$\therefore \iiint_V z\,dxdydz=\iiint_{V'} zr\,dr\,d\theta\,dz$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{a\cos\theta}rdr\int_0^{\sqrt{a^2-r^2}}zdz$$

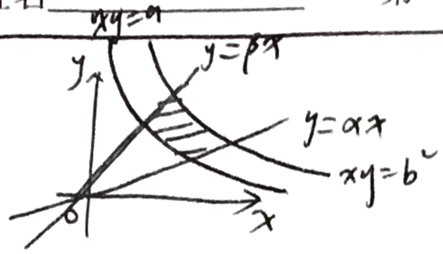
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{a\cos\theta}\left(\frac{a^2r^2}{4}-\frac{r^4}{8}\right)dr$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{a^4\cos^2\theta}{4}-\frac{a^4\cos^4\theta}{8}\right)d\theta=\frac{a^2}{4}\cdot\left(\frac{\theta}{2}+\frac{1}{4}\sin2\theta\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}-\frac{a^4}{8}\left(\frac{3}{8}\theta+\frac{\sin4\theta}{32}+\frac{\sin2\theta}{4}\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\frac{5a^4}{64}\pi$$

15.

积分域在 xoy 平面投影: D




$$原式 = \iint_D xy\,dx\,dy \int_{\frac{x^2+y^2}{n}}^{\frac{x^2+y^2}{m}} z\,dz$$

$$= \frac{n^2-m^2}{2m^2n^2} \iint_D xy\cdot(x^2+y^2)^2\,dx\,dy$$

令 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$ $\quad 原式 = \frac{n^2-m^2}{2n^2m^2}\iint_D r^7 \sin\theta\cos\theta\,dr\,d\theta$

$$= \frac{n^2-m^2}{2n^2m^2}\int_{\arctan\alpha}^{\arctan\beta} \sin\theta\cos\theta\,d\theta \int_{\frac{a}{\sqrt{\sin\theta\cos\theta}}}^{\frac{b}{\sqrt{\sin\theta\cos\theta}}} r^7\,dr$$

$$= \frac{(n^2-m^2)(b^8-a^8)}{16n^2m^2}\int_{\arctan\alpha}^{\arctan\beta} \frac{1}{\sin^3\theta\cos^3\theta}\,d\theta$$

$$\because \int\frac{d\theta}{\sin^3\theta\cos^3\theta} = \int\frac{d\sin\theta}{\sin^3\theta(1-\sin^2\theta)^2} = \int\frac{dx}{x^3(x-1)^2(x+1)^2} = \int\frac{dx}{x^3}+2\int\frac{dx}{x}-\int\frac{dx}{x+1}-\frac{1}{4}\int\frac{dx}{(x+1)^2}-\int\frac{dx}{x-1}+\frac{1}{4}\int\frac{dx}{(x-1)^2}$$

$$= -\frac{1}{2x^2}+2\ln|x|-\ln|x+1|+\frac{1}{4(x+1)}-\ln|x-1|-\frac{1}{4(x-1)} = -\frac{1}{2x^2}+\ln\frac{x^2}{x^2-1}-\frac{1}{2(x^2-1)}+C$$

$$\therefore \int_{\arctan\alpha}^{\arctan\beta}\frac{d\theta}{\sin^3\theta\cos^3\theta} = \frac{\beta^2-\alpha^2}{2}+\frac{1}{2}\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)+2\ln\frac{\beta}{a}$$

$$\therefore 原式 = \frac{(n^2-m^2)(b^8-a^8)}{16m^2n^2}\left[\frac{\beta^2-\alpha^2}{2}+\frac{1}{2}\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)+2\ln\frac{\beta}{\alpha}\right]$$

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15. 令 $u=\dfrac{x^2+y^2}{z}\in(m,n)$ $\quad v=xy\in(a^2,b^2)$ $\quad w=\dfrac{y}{x}\in(\alpha,\beta)$

解II

$$\left|\frac{D(u,v,w)}{D(x,y,z)}\right|=\begin{vmatrix}\frac{2x}{z} & \frac{2y}{z} & -\frac{x^2+y^2}{z^2}\\ y & x & 0\\ -\frac{y}{x^2} & \frac{1}{x} & 0\end{vmatrix}=-\frac{x^2+y^2}{z^2}\cdot\frac{2y}{x} \qquad \therefore\left|\frac{D(x,y,z)}{D(u,v,w)}\right|=-\frac{xz^2}{2y(x^2+y^2)}$$

$\because y=wx$, $\therefore v=wx^2$ $\quad\therefore x^2=\dfrac{v}{w}$, $y^2=w^2x^2=vw$, $z=\dfrac{v+vw^2}{uw}$

$$\iiint_V xyz\,dxdydz=\iiint_{V'}xyz\cdot\frac{xz^2}{2y(x^2+y^2)}dudvdw$$

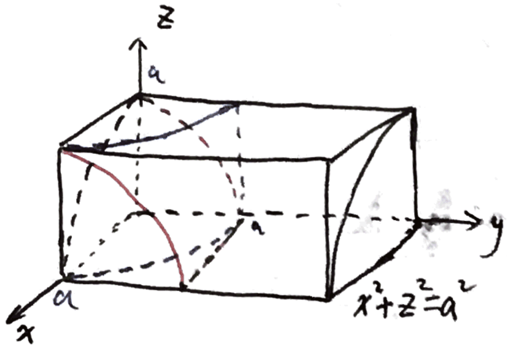
$$=\iiint_{V'}\frac{x^2z^3}{z(x^2+y^2)}dudvdw=\iiint_{V'}\frac{x^2z^2}{2u}dudwdv$$

$$=\iiint_{V'}\frac{v^3(1+w^2)^2}{2u^3w^3}=\int_m^n\frac{du}{2u^3}\int_{a^2}^{b^2}v^3dv\int_\alpha^\beta\frac{(1+w^2)^2}{w^3}dw$$

$$=\left(\frac{1}{4m^2}-\frac{1}{4n^2}\right)\left(\frac{b^8}{4}-\frac{a^8}{4}\right)\left(\frac{\beta^2-\alpha^2}{2}+\frac{1}{2}\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)+2\ln\frac{\beta}{\alpha}\right)$$

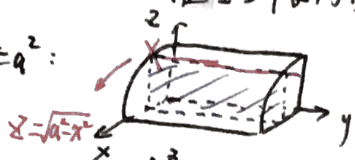
16.

(1)

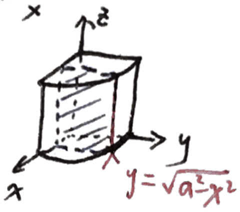



$x=x$ 平面去截 围成区域在 I 卦限内图象

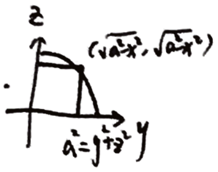
截 $x^2+z^2=a^2$:




或 $x^2+y^2=a^2$:

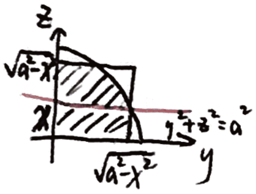



用平面 $\begin{cases} y=\sqrt{a^2-x^2} \\ z=\sqrt{a^2-x^2} \end{cases}$ 去截.




以 $a^2-x^2+a^2-x^2=a^2$, $x=\frac{a}{\sqrt{2}}$ 为分界.

$x\in[0,\frac{a}{\sqrt{2}}]$ 时



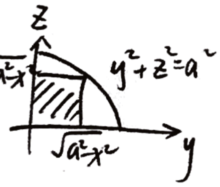

阴影为 $D_1$ 区域 $\iint_{D_1} dydz$

$$=\int_x^{\sqrt{a^2-x^2}} dz\int_0^{\sqrt{a^2-z^2}} dy + x\sqrt{a^2-x^2}$$

$$=\frac{z}{2}\sqrt{a^2-z^2}+\frac{a^2}{2}\arctan\frac{z}{\sqrt{a^2-z^2}}\Big|_x^{\sqrt{a^2-x^2}} + x\sqrt{a^2-x^2}$$

$$=x\sqrt{a^2-x^2}+\frac{a^2}{2}\left[\arctan\frac{\sqrt{a^2-x^2}}{x}-\arctan\frac{x}{\sqrt{a^2-x^2}}\right]$$

$x\in[\frac{a}{\sqrt{2}},a]$ 时




阴影为 $D_2$ 区域 $\iint_{D_2} dydz$

$$=a^2-x^2$$

$$\therefore \frac{V}{8}=\int_0^{\frac{a}{\sqrt{2}}} dx\iint_{D_1} dydz+\int_{\frac{a}{\sqrt{2}}}^{a} dx\iint_{D_2} dydz$$

$$=\int_0^{\frac{a}{\sqrt{2}}} x\sqrt{a^2-x^2}dx+\int_a^{\frac{a}{\sqrt{2}}}\frac{a^2}{2}\left[\arctan\frac{\sqrt{a^2-x^2}}{x}-\arctan\frac{x}{\sqrt{a^2-x^2}}\right]dx+\int_{\frac{a}{\sqrt{2}}}^{a}(a^2-x^2)dx$$

$$=-\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}\Big|_0^{\frac{a}{\sqrt{2}}}-2\sqrt{a^2-x^2}\Big|_0^{\frac{a}{\sqrt{2}}}-x\left(\arctan\frac{x}{\sqrt{a^2-x^2}}-\arctan\frac{\sqrt{a^2-x^2}}{x}\right)\Big|_0^{\frac{a}{\sqrt{2}}}+a^2x-\frac{x^3}{3}\Big|_{\frac{a}{\sqrt{2}}}^{a}$$

$$=(2-\sqrt{2})a^3$$

$$\therefore V=8(2-\sqrt{2})a^3$$

# 南開大學 作业纸

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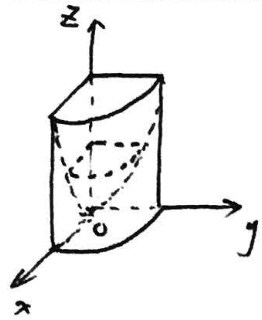
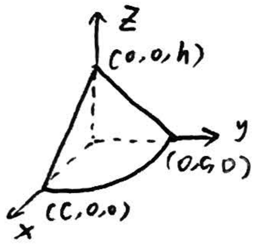
(2) 积分域在Ⅰ封限内:




作变换 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$$\therefore V = \pi a^2 \cdot h - \int_0^a dr \int_0^{2\pi} d\theta \int_{\frac{hr^2}{a^2}}^{h} r\,dz = \pi a^2 h - 2\pi \int_0^a \left(hr - \frac{hr^3}{a^2}\right) dr = \frac{\pi a^2 h}{2}$$

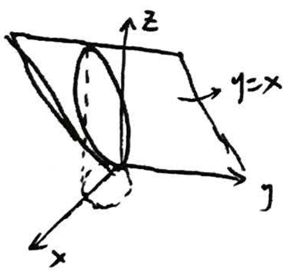
(3) 积分域在Ⅰ封限内:



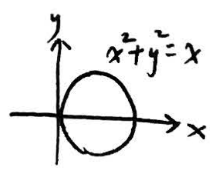

作变换 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$$\therefore V = \int_0^c r\,dr \int_0^{2\pi} d\theta \int_0^{\frac{h(c-r)}{c}} dz = 2\pi \int_0^c \left(hr - \frac{hr^2}{c}\right) dr = \frac{\pi h c^2}{3}$$

(4) 积分域:




xoy 投影:




作变换 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{\cos\theta} r\,dr \int_{r^2}^{r\cos\theta} dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{\cos\theta} (r^2\cos\theta - r^3)\,dr = \frac{1}{12} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\,d\theta$$

$$\because \int \cos^4\theta\,d\theta = \int \frac{(1+\cos 2\theta)^2}{4} d\theta = \frac{1}{4}\int (\cos^2 2\theta + 2\cos 2\theta + 1)\,d\theta = \frac{1}{4}\int \left(\frac{\cos 4\theta}{2} + 2\cos 2\theta + \frac{3}{2}\right) d\theta$$

$$= \frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta + C$$

$$\therefore V = \frac{1}{12} \cdot \frac{3}{8}\pi = \frac{\pi}{32}$$

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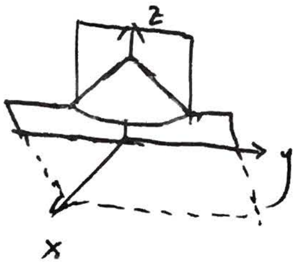
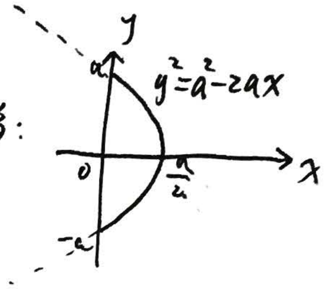
(5)



积分域在 $xoy$ 投影:




$$V=\int_{-a}^{a}dy\int_{0}^{\frac{a^2-y^2}{2a}}dx\int_{x}^{a-\sqrt{x^2+y^2}}dz=\int_{-a}^{a}dy\int_{0}^{\frac{a^2-y^2}{2a}}(a-x-\sqrt{x^2+y^2})dx$$

$$\because \int_{0}^{\frac{a^2-y^2}{2a}}\cdots=\left(ax-\frac{x^2}{2}-\frac{x}{2}\sqrt{x^2+y^2}-\frac{y^2}{2}\ln(x+\sqrt{x^2+y^2})\right)\Big|_{0}^{\frac{a^2-y^2}{2a}}$$

$$=\frac{a^2-y^2}{2}-\frac{(a^2-y^2)^2+(a^2-y^2)(a^2+y^2)}{8a^2}-\frac{1}{2}y^2\ln a+\frac{y^2\ln|y|}{2}=\frac{a^2-y^2}{4}-\frac{y^2}{2}\ln a+\frac{y^2}{2}\ln|y|$$

$$\therefore V=\int_{-a}^{a}\left(\frac{a^2-y^2}{4}-\frac{y^2}{2}\ln a+\frac{y^2}{2}\ln|y|\right)dy$$

$$=\left(\frac{a^2y}{4}-\frac{y^3}{12}-\frac{\ln a}{6}y^3-\frac{y^3}{18}\right)\Big|_{-a}^{a}+\frac{y^3}{6}\ln|y|\Big|_{-a}^{a}$$

$$=\frac{a^3}{2}-\frac{a^3}{6}-\frac{a^3}{9}$$

$$=\frac{2}{9}a^3$$