
CHAPTER 1

2012-2013 学年微积分（一）（上）期中考试

1 基本计算题 (每小题 5 分, 共 60 分)

1. 计算极限 $l = \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+\sqrt{2}} + \cdots + \frac{1}{n+\sqrt{n}} \right)$.

2. 计算极限 $l = \lim_{x \rightarrow \pi^+} \frac{\sqrt{1 + \cos x}}{\sin x}$.

3. 计算极限 $l = \lim_{x \rightarrow 0} \left(\frac{\ln(1+x)}{x} \right)^{\frac{1}{e^x - 1}}$.

4. 计算极限 $l = \lim_{x \rightarrow 0} \frac{\sqrt{1+2x} - 1 - x}{\sin x^2}$.

5. 设 $f(x) = \frac{(x-1)(x-2)\cdots(x-100)}{(x+1)(x+2)\cdots(x+100)}$, 求 $f'(1)$.

6. 设 $f(x) = \ln \sqrt{x + \sqrt{x + \sqrt{x + 1}}}$, 求 $f'(0)$.

7. 设 $y = y(x)$ 由 $xe^{f(y)} = x^y \ln 3$ 确定, $f(y)$ 可导, 且 $f'(y) \neq \ln x$, 求 dy .

8. 设 $y = y(x)$ 由参数方程 $\begin{cases} y = t^m \\ x = \ln 2t, \end{cases}$ 给出, 计算 $\left. \frac{d^n y}{dx^n} \right|_{t=1}$.

9. 设当 $x \rightarrow 0$ 时, $u = \sqrt[3]{1+x^2} \sqrt[3]{1-x} - 1 \sim cx^k$, 求 c, k 的值.

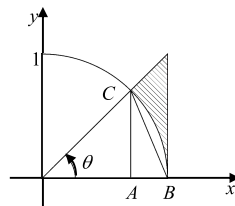
10. 设 $g(x) = \begin{cases} x \arctan \frac{1}{x}, & x < 0, \\ bx, & x \geq 0, \end{cases}$ 在 $x=0$ 处可导, $f(x) = \sin x$. 求 b 以及 $\left. \frac{d}{dx} f(g(x)) \right|_{x=0}$.

11. 设 $f(x) = \frac{3x}{2x^2 + x - 1}$, 计算 $f^{(n)}(0)$.

12. 设 $f(x) = \lim_{n \rightarrow \infty} \frac{x^n}{1 + x^n}$, 求出其所有间断点, 并说明间断点的类型.

2 综合题 (每小题 7 分, 共 28 分)

13. 如图所示, 在单位圆内, 当 $\theta \rightarrow 0$ 时, 证明三角形 ABC 的面积 $a(\theta)$ 与阴影部分的面积 $b(\theta)$ 是同阶无穷小.



14. 证明当 n 充分大时, $(\sqrt[n]{n} - 1)^{\frac{1}{n}} \left(\sqrt{1 + \frac{1}{n^2}} - 1 \right) < \frac{1}{n^2}$.

15. 若 $\lim_{n \rightarrow \infty} x_n = a$, $\lim_{n \rightarrow \infty} y_n = b$, 且 $a < b$, 依据极限定义证明当 n 充分大时, $x_n < y_n$.

16. 设物体 P 沿抛物线 $x = y^2 (y > 0)$ 自原点向右移动, 与原点的距离为 r . 设其水平速度 $\frac{dx}{dt}$ 保持为常量 A .

(1) 计算 $\frac{dr}{dt}$.

(2) 随着物体的移动, $\frac{dr}{dt}$ 是逐渐变大还是逐渐变小或者忽大忽小?

(3) 计算 $\frac{dr}{dt}$ 的最终极限.

3 证明题 (每小题 6 分, 共 12 分)

17. 设 $f(x)$ 在 $[a, b]$ 上连续, 在 (a, b) 内可导, 且 $f'(x) \neq 0$. 试证存在 $\xi, \eta \in (a, b)$, 使得

$$\frac{f'(\xi)}{f'(\eta)} = \frac{e^b - e^a}{b - a} e^{-\eta}.$$

18. 设 $f(x)$ 在闭区间 $[0, 1]$ 上连续, $f(0) = f(1)$, 证明存在 $x_0 \in [0, 1]$, 使得 $f(x_0) = f\left(x_0 + \frac{1}{4}\right)$.

CHAPTER 2

2012-2013 学年微积分（一）（上）期中考试参考答案

1 基本计算题 (每小题 5 分, 共 60 分)

1. **Solution.** $\frac{n}{n+\sqrt{n}} < \frac{1}{n+1} + \frac{1}{n+\sqrt{2}} + \cdots + \frac{1}{n+\sqrt{n}} < \frac{n}{n+1}.$

因为 $\lim_{n \rightarrow \infty} \frac{n}{n+\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$, 由夹逼定理得 $l = 1$.

2. **Solution.**

$$\begin{aligned} l &= \lim_{x \rightarrow \pi^+} \frac{\sqrt{2} \cos^2 \frac{x}{2}}{\sin x} = \lim_{x \rightarrow \pi^+} \frac{\sqrt{2} |\cos \frac{x}{2}|}{\sin x} \\ &= \lim_{x \rightarrow \pi^+} \frac{-\sqrt{2} \cos \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} \\ &= -\frac{\sqrt{2}}{2}. \end{aligned}$$

3. **Solution.**

$$\begin{aligned} l &= \exp \left\{ \lim_{x \rightarrow 0} \frac{1}{e^x - 1} \left[\frac{\ln(1+x)}{x} - 1 \right] \right\} \\ &= \exp \left[\lim_{x \rightarrow 0} \frac{\ln(1+x) - x}{x^2} \right] = \exp \left[\lim_{x \rightarrow 0} \frac{x - \frac{x^2}{2} + o(x^2) - x}{x^2} \right] \\ &= \frac{1}{\sqrt{e}}. \end{aligned}$$

4. **Solution.**

$$\begin{aligned} l &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+2x} - 1 - x)(\sqrt{1+2x} + 1 + x)}{x^2(\sqrt{1+2x} + 1 + x)} = \lim_{x \rightarrow 0} \frac{1 + 2x - (1+x)^2}{x^2(\sqrt{1+2x} + 1 + x)} \\ &= \lim_{x \rightarrow 0} \frac{1 + 2x - 1 - 2x - x^2}{x^2(\sqrt{1+2x} + 1 + x)} = -\frac{1}{2}. \end{aligned}$$

5. Solution.

$$\begin{aligned}
 f'(1) &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} \\
 &= \lim_{x \rightarrow 1} \frac{(x-1)(x-2) \cdots (x-100)}{(x+1)(x+2) \cdots (x+100)(x-1)} \\
 &= \lim_{x \rightarrow 1} \frac{(x-2) \cdots (x-100)}{(x+1)(x+2) \cdots (x+100)} \\
 &= \frac{(-1)^{99} \cdot 99!}{101!} = -\frac{1}{10100}.
 \end{aligned}$$

6. Solution.

$$\begin{aligned}
 f'(x) &= \left[\frac{1}{2} \ln \left(x + \sqrt{x + \sqrt{x+1}} \right) \right]' \\
 &= \frac{1}{2 \left(x + \sqrt{x + \sqrt{x+1}} \right)} \left(x + \sqrt{x + \sqrt{x+1}} \right)' \\
 &= \frac{1}{2 \left(x + \sqrt{x + \sqrt{x+1}} \right)} \left[1 + \frac{1}{2\sqrt{x + \sqrt{x+1}}} \left(1 + \frac{1}{2\sqrt{x+1}} \right) \right].
 \end{aligned}$$

将 $x=0$ 代入上式得 $f'(0) = \frac{1}{2} \left(1 + \frac{1}{2} \cdot \frac{3}{2} \right) = \frac{7}{8}$.

7. Solution. 取对数, 得 $\ln x + f(y) = y \ln x + \ln \ln 3$. 方程两边微分得

$$\frac{1}{x} dx + f'(y) dy = \ln x dy + y \cdot \frac{1}{x} dx.$$

整理得 $dy = \frac{y-1}{x[f'(y) - \ln x]} dx$.

8. Solution. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{mt^{m-1}}{\frac{2}{t}} = mt^m,$

$$\frac{d^2 y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx} = (mt^m)'_t \cdot \frac{1}{\frac{2}{t}} = m^2 t^m.$$

设 $\frac{d^k y}{dx^k} = m^k t^m$, 则 $\frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dt} \left(\frac{d^k y}{dx^k} \right) \cdot \frac{dt}{dx} = (m^k t^m)' \cdot \frac{1}{\frac{2}{t}} = m^{k+1} t^m.$

由数学归纳法得 $\frac{d^n y}{dx^n} = m^n t^m$, 所以 $\left. \frac{d^n y}{dx^n} \right|_{t=1} = m^n$.

9. Solution.

$$u = \sqrt[3]{1-x+x^2-x^3} - 1 = 1 + \frac{1}{3}(-x+x^2-x^3) + o(-x+x^2-x^3) - 1 = -\frac{1}{3}x + o(x).$$

所以 $u \sim -\frac{1}{3}x$, 即 $c = -\frac{1}{3}$, $k = 1$.

10. Solution. $g(0) = 0$, 因 $g'_-(0) = \lim_{x \rightarrow 0^-} \frac{g(x) - g(0)}{x} = \lim_{x \rightarrow 0^-} \arctan \frac{1}{x} = -\frac{\pi}{2} = g'_+(0) = b$, 所以 $b = -\frac{\pi}{2}$.

$$\left. \frac{d}{dx} f(g(x)) \right|_{x=0} = f'(g(0)) \cdot g'(0) = \cos 0 \cdot b = -\frac{\pi}{2}.$$

11. **Solution.** $f(x) = \frac{1}{2x-1} + \frac{1}{x+1}$, 所以

$$f^{(n)}(0) = \frac{(-1)^n \cdot n! \cdot 2^n}{(2x-1)^{n+1}} \Big|_{x=0} + \frac{(-1)^n \cdot n!}{(x+1)^{n+1}} \Big|_{x=0} = n! [(-1)^n - 2^n].$$

12. **Solution.** 当 $|x| > 1$ 时, $x^n \rightarrow \infty$, 所以 $f(x) = \lim_{n \rightarrow \infty} \frac{x^n}{1+x^n} = 1$;

当 $x = 1$ 时, $f(x) = \frac{1}{2}$; 当 $x = -1$ 时, $f(x)$ 不存在;

当 $|x| < 1$ 时, $x^n \rightarrow 0$, 所以 $f(x) = \lim_{n \rightarrow \infty} \frac{x^n}{1+x^n} = 0$.

$$\text{综上所述, } f(x) = \begin{cases} 1, & |x| > 1, \\ \frac{1}{2}, & x = 1, \\ \text{不存在}, & x = -1, \\ 0, & |x| < 1. \end{cases}$$

$f(x)$ 的间断点为 $x = -1$ 和 $x = 1$.

因为 $f(1^-) = 0$, $f(1^+) = 1$, $f(-1^-) = 1$, $f(-1^+) = 0$, 所以 $x = 1$ 和 $x = -1$ 都是跳跃间断点.

2 综合题 (每小题 7 分, 共 28 分)

13. **Solution.** 由几何关系, $a(\theta) = \frac{1}{2} \sin \theta (1 - \cos \theta)$, $b(\theta) = \frac{1}{2} \tan \theta - \frac{1}{2} \theta$.

当 $\theta \rightarrow 0$ 时, $a(\theta) \rightarrow 0$, $b(\theta) \rightarrow 0$, 所以 $a(\theta)$ 和 $b(\theta)$ 都是无穷小. 计算

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{a(\theta)}{b(\theta)} &= \lim_{\theta \rightarrow 0} \frac{\sin \theta (1 - \cos \theta)}{\tan \theta - \theta} \\ &= \frac{1}{2} \lim_{\theta \rightarrow 0} \frac{\theta^3}{\tan \theta - \theta} = \frac{1}{2} \lim_{\theta \rightarrow 0} \frac{3\theta^2}{\sec^2 \theta - 1} \\ &= \frac{3}{2} \lim_{\theta \rightarrow 0} \frac{\theta^2}{\tan^2 \theta} = \frac{3}{2}. \end{aligned}$$

所以当 $\theta \rightarrow 0$ 时, $a(\theta)$ 与 $b(\theta)$ 是同阶无穷小.

14. **Solution.** 计算

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{(\sqrt{x} - 1)^{\frac{1}{x}} \left(\sqrt{1 + \frac{1}{x^2}} - 1 \right)}{\frac{1}{x^2}} &= \lim_{x \rightarrow +\infty} (\sqrt{x} - 1)^{\frac{1}{x}} \cdot \lim_{x \rightarrow +\infty} \frac{\frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)}{\frac{1}{x^2}} \\ &= \frac{1}{2} \lim_{x \rightarrow +\infty} (\sqrt{x} - 1)^{\frac{1}{x}} = \frac{1}{2} \lim_{x \rightarrow +\infty} \left(e^{\frac{\ln x}{x}} - 1 \right)^{\frac{1}{x}} \\ &= \frac{1}{2} \exp \left[\lim_{x \rightarrow +\infty} \frac{\ln \left(e^{\frac{\ln x}{x}} - 1 \right)}{x} \right] \\ &= \frac{1}{2} \exp \left[\lim_{x \rightarrow +\infty} \frac{e^{\frac{\ln x}{x}} \cdot \frac{1 - \ln x}{x^2}}{e^{\frac{\ln x}{x}} - 1} \right] = \frac{1}{2} \exp \left[\lim_{x \rightarrow +\infty} \frac{e^{\frac{\ln x}{x}} (1 - \ln x)}{x \ln x} \right] \\ &= \frac{1}{2} \exp \left[\lim_{x \rightarrow +\infty} \frac{1 - \ln x}{x \ln x} \right] = \frac{1}{2} \exp \left[\lim_{x \rightarrow +\infty} \frac{\frac{1}{\ln x} - 1}{x} \right] = \frac{1}{2}. \end{aligned}$$

$$\text{所以 } \lim_{n \rightarrow \infty} \frac{(\sqrt[n]{n} - 1)^{\frac{1}{n}} \left(\sqrt{1 + \frac{1}{n^2}} - 1 \right)}{\frac{1}{n^2}} = \frac{1}{2} < 1,$$

由极限的保号性, 当 n 充分大时, 有 $(\sqrt[n]{n} - 1)^{\frac{1}{n}} \left(\sqrt{1 + \frac{1}{n^2}} - 1 \right) < \frac{1}{n^2}$.

15. **Solution.** 对于 $\varepsilon = \frac{b-a}{2} > 0$,

由极限的定义, 存在 N_1 使得当 $n > N_1$ 时, $|x_n - a| < \varepsilon$; 存在 N_2 使得当 $n > N_2$ 时, $|y_n - b| < \varepsilon$.

设 $N = \max\{N_1, N_2\}$, 则当 $n > N$ 时, $x_n < a + \varepsilon = \frac{a+b}{2} < b - \varepsilon < y_n$.

16. **Solution.** (1) 由题设 $r = \sqrt{x^2 + y^2} = \sqrt{x^2 + x}$, 所以 $\frac{dr}{dt} = \frac{2x+1}{2\sqrt{x^2+x}} \cdot \frac{dx}{dt} = \frac{(2x+1)A}{2\sqrt{x^2+x}}$.

(2)

$$\begin{aligned} \frac{d^2r}{dt^2} &= \frac{d}{dx} \left[\frac{(2x+1)A}{2\sqrt{x^2+x}} \right] \cdot \frac{dx}{dt} \\ &= \frac{1}{2} \cdot \frac{2\sqrt{x^2+x} - \frac{(2x+1)^2}{2\sqrt{x^2+x}}}{x^2+x} \cdot A^2 \\ &= \frac{1}{2} \cdot \frac{4(x^2+x) - (2x+1)^2}{2(x^2+x)\sqrt{x^2+x}} \cdot A^2 = -\frac{A^2}{(x^2+x)\sqrt{x^2+x}} < 0. \end{aligned}$$

所以 $\frac{dr}{dt}$ 逐渐变小.

$$(3) \lim_{x \rightarrow +\infty} \frac{dr}{dt} = \lim_{x \rightarrow +\infty} \frac{(2x+1)A}{2\sqrt{x^2+x}} = A.$$

3 证明题 (每小题 6 分, 共 12 分)

17. **Proof.** 函数 $f(x)$ 和函数 $y = e^x$ 均在 $[a, b]$ 上连续, 在 (a, b) 内可导, 且 $e^x \neq 0$.

由 Cauchy 中值定理, 存在 $\eta \in (a, b)$ 使得

$$\frac{f(b) - f(a)}{e^b - e^a} = \frac{f'(\eta)}{e^\eta}.$$

对函数 $f(x)$ 应用 Lagrange 中值定理, 存在 $\xi \in (a, b)$ 使得 $f(b) - f(a) = f'(\xi)(b - a)$,

代入上式即得 $\frac{f'(\xi)}{f'(\eta)} = \frac{e^b - e^a}{b - a} e^{-\eta}$.

18. **Proof.** 令 $F(x) = f(x) - f\left(x + \frac{1}{4}\right)$, 显然 $F(x)$ 在 $\left[0, \frac{3}{4}\right]$ 上连续.

注意到

$$\begin{aligned} \frac{F(0) + F\left(\frac{1}{4}\right) + F\left(\frac{2}{4}\right) + F\left(\frac{3}{4}\right)}{4} &= \frac{1}{4} \left[f(0) - f\left(\frac{1}{4}\right) + f\left(\frac{1}{4}\right) - f\left(\frac{2}{4}\right) + f\left(\frac{2}{4}\right) - f\left(\frac{3}{4}\right) + f\left(\frac{3}{4}\right) - f(1) \right] \\ &= \frac{1}{4} [f(0) - f(1)] = 0. \end{aligned}$$

由介值定理, 存在 $x_0 \in \left[0, \frac{3}{4}\right] \subset [0, 1]$, 使得 $F(x_0) = 0$, 即 $f(x_0) = f\left(x_0 + \frac{1}{4}\right)$.