Third order geometric nonlinearity analysis of a double-bar Biot truss

(solved by four different numerical methods)

Input data

Strut length - $l = 2\,\mathrm{m}$

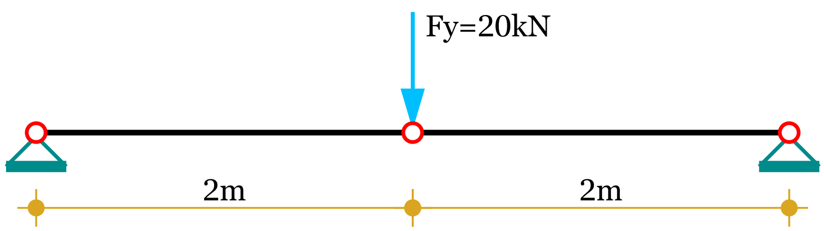
Material - steel. Modulus of elasticity - $E = 210\,\mathrm{GPa}$

Cross section - circular with diameter $\Phi = 20\,\mathrm{mm}$

Area - $A = \dfrac{\pi \cdot \Phi^2}{4} = \dfrac{3.14 \cdot (20\,\mathrm{mm})^2}{4} = 3.14\,\mathrm{cm}^2$

Axial stiffness - $EA = E \cdot A = 210\,\mathrm{GPa} \cdot 3.14\,\mathrm{cm}^2 = 65973.4\,\mathrm{kN}$

Vertical force - $F_y = 20\,\mathrm{kN}$



Solution

Because of the symmetry, the horizontal displacement in the middle is $u = 0\,\mathrm{m}$.

The vertical displacement is the only unknown - v = ?

Since the system is linearly unstable, we use 3-rd order geometric nonlinearity theory for the solution. The equilibrium equations are then derived for the deformed state of the structure, as follows:

Length and elongation in deformed state

$l'(v) = \sqrt{(l + u)^2 + v^2}$, $\Delta l(v) = l'(v) - l$

Horizontal reaction - $F_x(v) = EA \cdot \dfrac{\Delta l(v)}{l} \cdot \dfrac{l + u}{l'(v)}$

Vertical reaction - $F_y(v) = EA \cdot \dfrac{\Delta l(v)}{l} \cdot \dfrac{v}{l'(v)}$

Vertical reaction derivative - $F'_{yv}(v) = EA \cdot \left(\dfrac{1}{l} - \dfrac{(l + u)^2}{l'(v)^3} \right)$

1. Fixed point iteration method

Relative strain - $\varepsilon = \dfrac{F_y}{2 \cdot EA} = \dfrac{20\,\mathrm{kN}}{2 \cdot 65973.4\,\mathrm{kN}} = 0.000152$

Relative precision - $\delta_{max} = 10^{-4} = 0.0001$

Initial value - $v_0 = 200\,\mathrm{mm}$

We express the unknown vertical displacement at the middle joint as a function of the vertical force:

$$v = \sqrt{\frac{1}{\left(\frac{1}{l} - \frac{\varepsilon}{v_0}\right)^2} - (l + u)^2} = \sqrt{\frac{1}{\left(\frac{1}{2\,\mathrm{m}} - \frac{0.000152}{200\,\mathrm{mm}}\right)^2} - (2\,\mathrm{m} + 0\,\mathrm{m})^2} = 110.24\,\mathrm{mm}$$

After calculating the above expression iteratively $n = 13$ times, we get:

$v = 134.51\,\mathrm{mm}$

Relative error - $\delta = \frac{|v - v_0|}{|v|} = \frac{|134.51\,\mathrm{mm} - 134.5\,\mathrm{mm}|}{|134.51\,\mathrm{mm}|} = 7.6\times10^{-5}$

2. Newton-Raphson's method

Initial value - $v_0 = 200\,\mathrm{mm}$

We repeatedly calculate the following expression:

$$v = v_0 - \frac{2\cdot F_y(v_0) - F_y}{F'_{yv}(v_0)} = 200\,\mathrm{mm} - \frac{2\cdot F_y(200\,\mathrm{mm}) - 20\,\mathrm{kN}}{F'_{yv}(200\,\mathrm{mm})} = 106.93\,\mathrm{mm}$$

After $n = 4$ iterations we get: $v = 134.51\,\mathrm{mm}$

Relative error - $\delta = \frac{|v - v_0|}{|v|} = \frac{|134.51\,\mathrm{mm} - 134.51\,\mathrm{mm}|}{|134.51\,\mathrm{mm}|} = 9\times10^{-6}$

3. Secant method

Slope reduction factor - $\alpha = 1$

Initial value - $v_0 = 200\,\mathrm{mm}$

Force value - $F_{y0} = 2\cdot F_y(v_0) = 2\cdot F_y(200\,\mathrm{mm}) = 65.48\,\mathrm{kN}$

We calculate the first approximation using Newton-Raphson's method

$$v_1 = v_0 - \alpha\cdot\frac{F_{y0} - F_y}{2\cdot F'_{yv}(v_0)} = 200\,\mathrm{mm} - 1\cdot\frac{65.48\,\mathrm{kN} - 20\,\mathrm{kN}}{2\cdot F'_{yv}(200\,\mathrm{mm})} = 153.46\,\mathrm{mm}$$

Force value - $F_{y1} = 2\cdot F_y(v_1) = 2\cdot F_y(153.46\,\mathrm{mm}) = 29.67\,\mathrm{kN}$

The next approximation is evaluated by the formula:

$$v_2 = v_1 - \alpha\cdot(F_{y1} - F_y)\cdot\frac{v_1 - v_0}{F_{y1} - F_{y0}} = 153.46\,\mathrm{mm} - 1\cdot(29.67\,\mathrm{kN} - 20\,\mathrm{kN})\cdot\frac{153.46\,\mathrm{mm} - 200\,\mathrm{mm}}{29.67\,\mathrm{kN} - 65.48\,\mathrm{kN}} = 140.89\,\mathrm{mm}$$

We continue the calculations iteratively until we reach convergence.

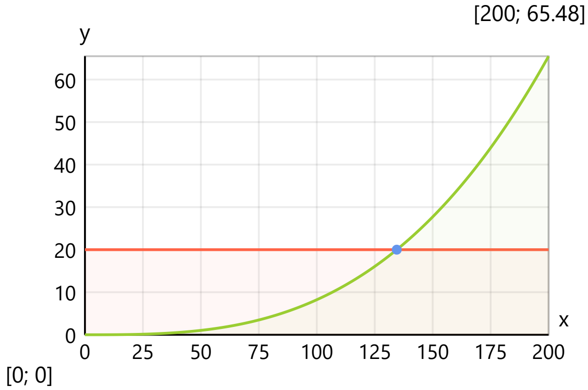
After $n = 4$ iterations we get: $v_2 = 134.51\,\mathrm{mm}$

Relative error - $\delta = \frac{|v_2 - v_1|}{|v_2|} = \frac{|134.51\,\mathrm{mm} - 134.51\,\mathrm{mm}|}{|134.51\,\mathrm{mm}|} = 1.57\times10^{-6}$

4. Solution with Calcpad (modified Anderson-Bjork's method)

$v = \$\mathrm{Root}\{2\cdot F_y(v) = F_y;\ v \in [0\,\mathrm{mm};\ 200\,\mathrm{m}]\} = 134.51\,\mathrm{mm}$

System behavior graph (force-displacement)



Results

Axial forces in bars - $N = \dfrac{\Delta l(v)}{l} \cdot EA = \dfrac{\Delta l(134.51\ \mathrm{mm})}{2\ \mathrm{m}} \cdot 65973.4\ \mathrm{kN} = 149.03\ \mathrm{kN}$

Rotation angle - $\alpha = \mathbf{atan2}(l;\ v) = \mathbf{atan2}(2\ \mathrm{m};\ 134.51\ \mathrm{mm}) = 3.85\ °$

Reactions in supports

Horizontal - $R_x = F_x(v) = F_x(134.51\ \mathrm{mm}) = 148.69\ \mathrm{kN} = N \cdot \mathbf{cos}(\alpha) = 149.03\ \mathrm{kN} \cdot \mathbf{cos}(3.85) = 148.69\ \mathrm{kN}$

Vertical - $R_y = F_y(v) = F_y(134.51\ \mathrm{mm}) = 10\ \mathrm{kN} = N \cdot \mathbf{sin}(\alpha) = 149.03\ \mathrm{kN} \cdot \mathbf{sin}(3.85) = 10\ \mathrm{kN}$