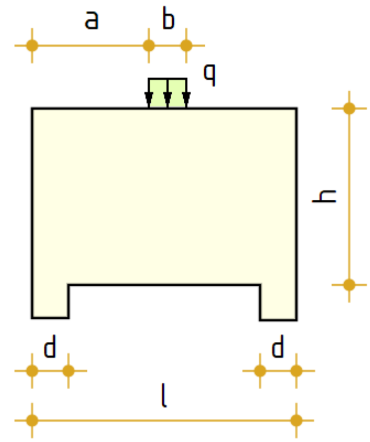
Finite Element Analysis of Deep Beam with Calcpad

Input data



Length - $l = 4$ m, Height - $h = 2$ m

Thickness - $t = 0.1$ m

Loads

Distributed load - $q = 100$ kN/m

Load length - $b = 0.8$ m

Load position - $a = \dfrac{l - b}{2} = \dfrac{4 - 0.8}{2} = 1.6$ m

Load function - $q(x) = q \cdot (x \geq a \; and \; x \leq a + b)$

Supports

Support length - $d = 0.4$ m

Support elastic stiffness - $k_s = 50000$ MN/m²

Support function - $s(x) = k_s \cdot t \cdot (x \leq d \; or \; x \geq l - d)$

Material properties

Modulus of elasticity - $E = 20000$ MPa

Poisson`s ratio - $\nu = 0.2$

Finite element mesh

We will use rectangular finite element with $n = 8$ DOFs

Number of elements along *l* and *h* - $n_l = 20$, $n_h = 10$

Total number of elements - $n_e = n_l \cdot n_h = 20 \cdot 10 = 200$

Total number of joints - $n_j = (n_l + 1) \cdot (n_h + 1) = (20 + 1) \cdot (10 + 1) = 231$

Element dimensions - $l_1 = \dfrac{l}{n_l} = \dfrac{4}{20} = 0.2$, $h_1 = \dfrac{h}{n_h} = \dfrac{2}{10} = 0.2$

Joint coordinates

$\vec{x}_j = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0.2 \;\; 0.2 \;\; 0.2 \;\; 0.2 \;\; 0.2 \;\; 0.2 \;\; 0.2 \;\; 0.2 \;\; 0.2 \;\; \ldots \;\; 4]$ m

$\vec{y}_j = [0 \;\; 0.2 \;\; 0.4 \;\; 0.6 \;\; 0.8 \;\; 1 \;\; 1.2 \;\; 1.4 \;\; 1.6 \;\; 1.8 \;\; 2 \;\; 0 \;\; 0.2 \;\; 0.4 \;\; 0.6 \;\; 0.8 \;\; 1 \;\; 1.2 \;\; 1.4 \;\; 1.6 \;\; \ldots \;\; 2]$ m

Joint numbers for elements

$$\mathbf{transp}(e_j) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & \cdots & 219 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & \cdots & 220 \\ 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & \cdots & 230 \\ 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 & \cdots & 231 \end{bmatrix}$$

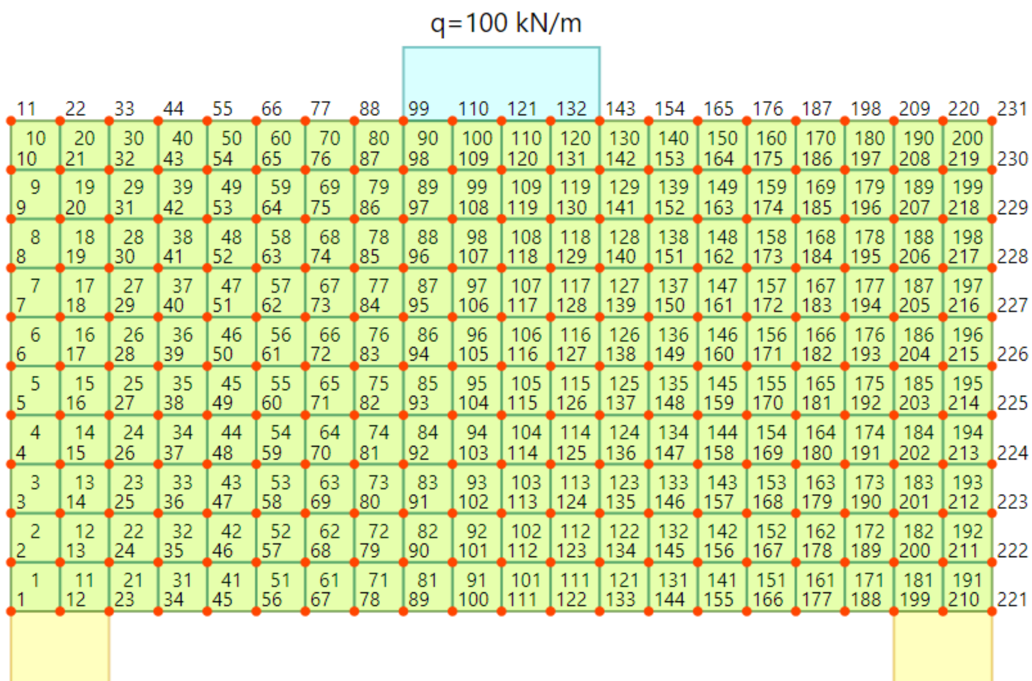
Coordinates of element centers

$$j_e(e) = \mathbf{row}(e_j; e),\quad x_c(e) = \frac{\mathbf{sum}\left(\mathbf{extract}\left(\vec{x}_j; j_e(e)\right)\right)}{4},\quad y_c(e)\frac{\mathbf{sum}\left(\mathbf{extract}\left(\vec{y}_j; j_e(e)\right)\right)}{4}$$

$\vec{x}_c$ = [0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.3 0.3 0.3 0.3 0.3 0.3 0.3 0.3 0.3 0.3 ... 3.9] m
$\vec{y}_c$ = [0.1 0.3 0.5 0.7 0.9 1.1 1.3 1.5 1.7 1.9 0.1 0.3 0.5 0.7 0.9 1.1 1.3 1.5 1.7 1.9 ... 1.9] m

Elements along the bottom (supported) edge - $\vec{e}_S = \mathbf{find}_{lt}(\vec{y}_c; h_1; 1) = \mathbf{find}_{lt}(\vec{y}_c; 0.2; 1) =$
[1 11 21 31 41 51 61 71 81 91 101 111 121 131 141 151 161 171 181 191]

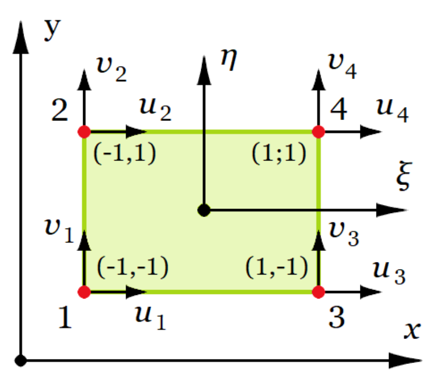
Elements along the top (loaded) edge - $\vec{e}_L = \mathbf{find}_{gt}(\vec{y}_c; h - h_1; 1) = \mathbf{find}_{gt}(\vec{y}_c; 2 - 0.2; 1) =$
[10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200]



Finite element formation

Shape functions

They are defined in relative coordinates to the center of the element:



$$\xi = \frac{2\cdot(x - \vec{x}_c)}{l_1} \in (-1;\ 1),\quad \eta = \frac{2\cdot(y - \vec{y}_c)}{h_1} \in (-1;\ 1)$$

Base functions - $\Phi_1(\xi) = \dfrac{1-\xi}{2}, \Phi_2(\xi) = \dfrac{1+\xi}{2}$

First derivatives - $\Phi'_1(\xi) = -\dfrac{1}{2}, \Phi'_2(\xi) = \dfrac{1}{2}$

Shape functions for joints at elements\` corners

$$N_1(\xi;\eta) = \Phi_1(\xi)\cdot\Phi_1(\eta)\ ,\ N_2(\xi;\eta) = \Phi_1(\xi)\cdot\Phi_2(\eta)$$

$$N_3(\xi;\eta) = \Phi_2(\xi)\cdot\Phi_1(\eta)\ ,\ N_4(\xi;\eta) = \Phi_2(\xi)\cdot\Phi_2(\eta)$$

Constitutive matrix (stress-strain relationship)

$$D = \frac{E \cdot t}{1 - \nu^2} \cdot \left[1; \nu; 0 \mid \nu; 1; 0 \mid 0; 0; \frac{1-\nu}{2}\right] = \frac{20000 \cdot 0.1}{1 - 0.2^2} \cdot \left[1; 0.2; 0 \mid 0.2; 1; 0 \mid 0; 0; \frac{1-0.2}{2}\right] =$$

$$\begin{bmatrix} 2083.33 & 416.67 & 0 \\ 416.67 & 2083.33 & 0 \\ 0 & 0 & 833.33 \end{bmatrix}$$

Strain-displacement matrix

$$B_1(j; \eta) = \frac{1}{l_1} \cdot \mathbf{take}(j; \text{-}\Phi_1(\eta); 0; \text{-}\Phi_2(\eta); 0; \Phi_1(\eta); 0; \Phi_2(\eta); 0)$$

$$B_2(j; \xi) = \frac{1}{h_1} \cdot \mathbf{take}\left(j; 0; \text{-}\Phi_1(\xi); 0; \Phi_1(\xi); 0; \text{-}\Phi_2(\xi); 0; \Phi_2(\xi)\right)$$

$$B_3(j; \xi; \eta) = \mathbf{take}\left(j; \frac{\text{-}\Phi_1(\xi)}{h_1}; \frac{\text{-}\Phi_1(\eta)}{l_1}; \frac{\Phi_1(\xi)}{h_1}; \frac{\text{-}\Phi_2(\eta)}{l_1}; \frac{\text{-}\Phi_2(\xi)}{h_1}; \frac{\Phi_1(\eta)}{l_1}; \frac{\Phi_2(\xi)}{h_1}; \frac{\Phi_2(\eta)}{l_1}\right)$$

$$B(j; \xi; \eta) = [B_1(j; \eta); B_2(j; \xi); B_3(j; \xi; \eta)]$$

The elements of the stiffness matrix will be calculated by using direct integration

$$K_{e,ij} = \frac{l_1 \cdot h_1}{4} \cdot \int_{-1}^{1} \int_{-1}^{1} B_i(\xi; \eta)^T \cdot D \cdot B_j(\xi; \eta) \, d\xi \, d\eta$$

Element stiffness matrix (above the main diagonal only)

$$BTDB_e(i; j; \xi; \eta) = \mathbf{transp}\left(B(i; \xi; \eta)\right) \cdot D \cdot B(j; \xi; \eta)$$

$$K_e(i; j) = \frac{l_1 \cdot h_1}{4} \cdot \int_{-1}^{1} \int_{-1}^{1} BTDB_e(i; j; \xi; \eta) \, d\xi \, d\eta$$

$$\$\text{Repeat}\left\{\$\text{Repeat}\left\{K_{e.i,j} = K_e(i; j); j = i \ldots n\right\}; i = 1 \ldots n\right\}$$

$$K_e = \begin{bmatrix} 972.22 & 312.5 & 69.44 & 104.17 & -555.56 & -104.17 & -486.11 & -312.5 \\ 0 & 972.22 & -104.17 & -555.56 & 104.17 & 69.44 & -312.5 & -486.11 \\ 0 & 0 & 972.22 & -312.5 & -486.11 & 312.5 & -555.56 & 104.17 \\ 0 & 0 & 0 & 972.22 & 312.5 & -486.11 & -104.17 & 69.44 \\ 0 & 0 & 0 & 0 & 972.22 & -312.5 & 69.44 & -104.17 \\ 0 & 0 & 0 & 0 & 0 & 972.22 & 104.17 & -555.56 \\ 0 & 0 & 0 & 0 & 0 & 0 & 972.22 & 312.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 972.22 \end{bmatrix} \text{MN/m}$$

Boundary conditions

Supports

Number of elements along the supported edge - $n_S = \mathbf{len}(\vec{e}_S) = 20$

Element's joint springs stiffness factors

$$K_{s,j}(e) = \frac{l_1}{2} \cdot \int_{-1}^{1} N_j(\xi; -1) \cdot s\left(\vec{x}_{c.e} + \frac{\xi \cdot l_1}{2}\right) d\xi\,, \quad j = 1\,, 3$$

Results for element 1

$$K_{s,1}(e) = \frac{l_1}{2} \cdot \int_{-1}^{1} N_1(\xi; -1) \cdot s\left(\vec{x}_{c.e} + \frac{\xi \cdot l_1}{2}\right) d\xi, \quad K_{s,1}(1) = 500\ \mathrm{MN/m}$$

$$K_{s,3}(e) = \frac{l_1}{2} \cdot \int_{-1}^{1} N_3(\xi; -1) \cdot s\left(\vec{x}_{c.e} + \frac{\xi \cdot l_1}{2}\right) d\xi, \quad K_{s,3}(1) = 500\mathrm{MN/m}$$

Loads

Number of elements along the loaded edge - $n_L = \mathbf{len}(\vec{e}_L) = 20$

Element load vector

$$F_j(e) = \frac{l_1}{2} \cdot \int_{-1}^{1} N_j(\xi; 1) \cdot q\left(\vec{x}_{c.e} + \frac{\xi \cdot l_1}{2}\right) d\xi\,, \quad j = 2\,, 4$$

Results for element 100

$$F_2(e) = \frac{l_1}{2} \cdot \int_{-1}^{1} N_2(\xi; 1) \cdot q\left(\vec{x}_{c.e} + \frac{\xi \cdot l_1}{2}\right) d\xi, \quad F_2(e) = F_2(100) = 10\ \mathrm{kN}$$

$$F_4(e) = \frac{l_1}{2} \cdot \int_{-1}^{1} N_4(\xi; 1) \cdot q\left(\vec{x}_{c.e} + \frac{\xi \cdot l_1}{2}\right) d\xi, \quad F_4(e) = F_4(100) = 10\ \mathrm{kN}$$

Solution

Global stiffness matrix - $K =$

$$\begin{bmatrix}
972.22 & 312.5 & 69.44 & 104.17 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
312.5 & 1472.22 & -104.17 & -555.56 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
69.44 & -104.17 & 1944.44 & 0 & 69.44 & 104.17 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
104.17 & -555.56 & 0 & 1944.44 & -104.17 & -555.56 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 69.44 & -104.17 & 1944.44 & 0 & 69.44 & 104.17 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 104.17 & -555.56 & 0 & 1944.44 & -104.17 & -555.56 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 69.44 & -104.17 & 1944.44 & 0 & 69.44 & 104.17 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 104.17 & -555.56 & 0 & 1944.44 & -104.17 & -555.56 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 69.44 & -104.17 & 1944.44 & 0 & 69.44 & 104.17 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 104.17 & -555.56 & 0 & 1944.44 & -104.17 & -555.56 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 69.44 & -104.17 & 1944.44 & 0 & 69.44 & 104.17 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 104.17 & -555.56 & 0 & 1944.44 & -104.17 & -555.56 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 69.44 & -104.17 & 1944.44 & 0 & 69.44 & 104.17 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 104.17 & -555.56 & 0 & 1944.44 & -104.17 & -555.56 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 69.44 & -104.17 & 1944.44 & 0 & 69.44 & 104.17 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 104.17 & -555.56 & 0 & 1944.44 & -104.17 & -555.56 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 69.44 & -104.17 & 1944.44 & 0 & 69.44 & 104.17 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 104.17 & -555.56 & 0 & 1944.44 & -104.17 & -555.56 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 69.44 & -104.17 & 1944.44 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 104.17 & -555.56 & 0 & 1944.44 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 972.22
\end{bmatrix}$$

Global load vector

$$\vec{F} = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots\ 0]$$

$$\mathbf{sum}(\vec{F}) = 80\ \mathrm{kN}$$

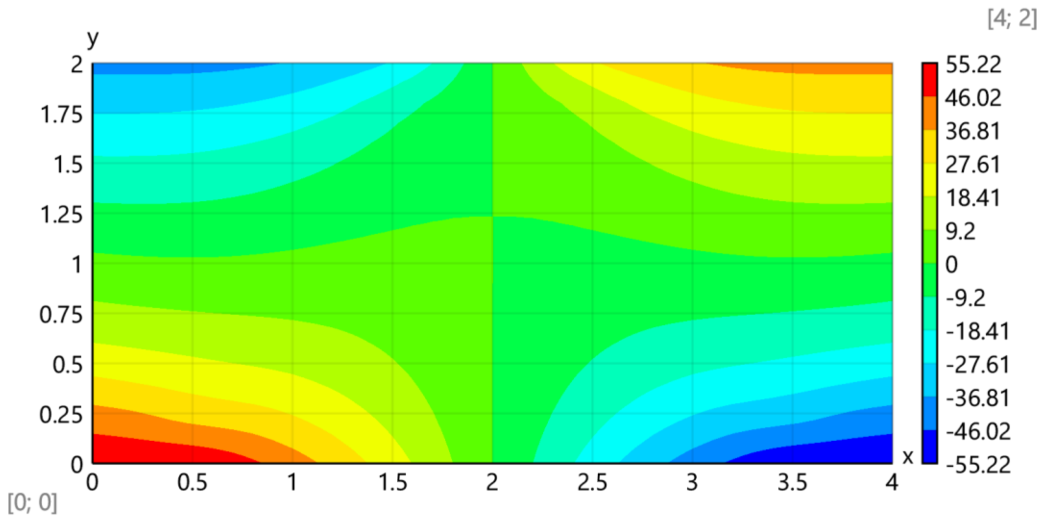
Solution of the system of equations - $\vec{Z} = \mathbf{clsolve}(K; \vec{F}) =$

$$[0.0543\quad 0.00979\quad 0.0429\quad 0.0144\quad 0.0296\quad 0.0215\quad 0.0185\quad 0.0288\quad 0.00958\quad 0.0348\quad 0.00188\ \ldots$$
$$\ldots\ 0.0391\quad -0.00543\quad 0.0419\quad -0.013\quad 0.0433\quad -0.0212\quad 0.0438\quad -0.0302\quad 0.0438\quad \ldots\quad 0.0438]\ \mathrm{mm}$$

Results

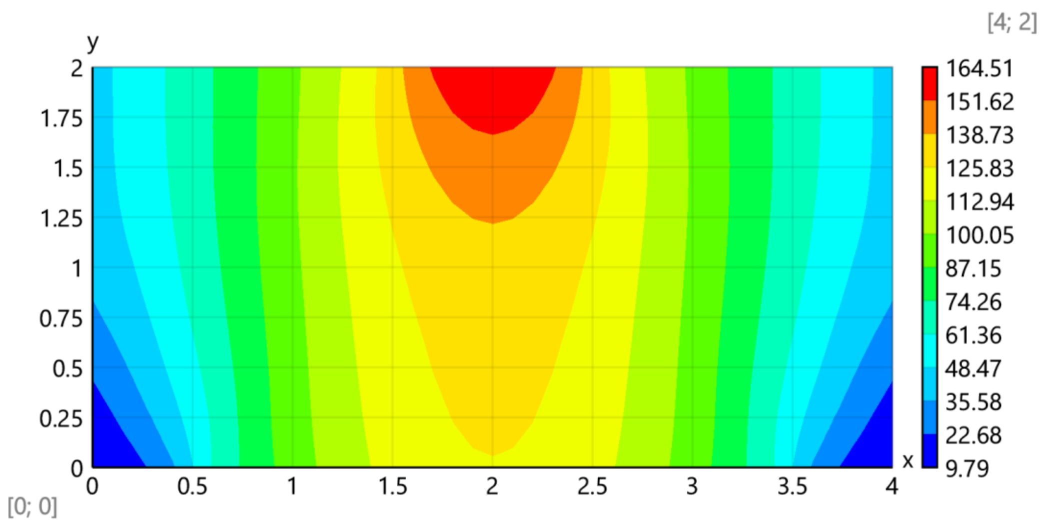
Horizontal joint displacements, $\cdot 10^{-3}$mm - $\mathrm{transp}(u) =$

$$\begin{bmatrix}
54.3 & 54.39 & 55.22 & 52.97 & 47.42 & 41.13 & 34.17 & 26.47 & 18.09 & 9.19 & 0 & -9.19 & -18.09 & -26.47 & -34.17 & -41.13 & -47.42 & -52.97 & -55.22 & -54.39 & -54.3 \\
42.91 & 40.91 & 37.67 & 35.3 & 33.21 & 29.57 & 24.88 & 19.41 & 13.31 & 6.77 & 0 & -6.77 & -13.31 & -19.41 & -24.88 & -29.57 & -33.21 & -35.3 & -37.67 & -40.91 & -42.91 \\
29.56 & 27.77 & 25.68 & 23.77 & 22.13 & 20.11 & 17.25 & 13.65 & 9.46 & 4.84 & 0 & -4.84 & -9.46 & -13.65 & -17.25 & -20.11 & -22.13 & -23.77 & -25.68 & -27.77 & -29.56 \\
18.49 & 17.2 & 15.95 & 14.82 & 13.83 & 12.71 & 11.16 & 9.04 & 6.37 & 3.3 & 0 & -3.3 & -6.37 & -9.04 & -11.16 & -12.71 & -13.83 & -14.82 & -15.95 & -17.2 & -18.49 \\
9.58 & 8.65 & 7.94 & 7.43 & 7.1 & 6.79 & 6.25 & 5.31 & 3.89 & 2.07 & 0 & -2.07 & -3.89 & -5.31 & -6.25 & -6.79 & -7.1 & -7.43 & -7.94 & -8.65 & -9.58 \\
1.88 & 1.27 & 0.925 & 0.907 & 1.19 & 1.63 & 2.03 & 2.14 & 1.81 & 1.05 & 0 & -1.05 & -1.81 & -2.14 & -2.03 & -1.63 & -1.19 & -0.907 & -0.925 & -1.27 & -1.88 \\
-5.43 & -5.76 & -5.82 & -5.44 & -4.6 & -3.41 & -2.08 & -0.893 & -0.128 & 0.116 & 0 & -0.116 & 0.128 & 0.893 & 2.08 & 3.41 & 4.6 & 5.44 & 5.82 & 5.76 & 5.43 \\
-12.99 & -13.1 & -12.92 & -12.22 & -10.93 & -9.05 & -6.74 & -4.34 & -2.31 & -0.915 & 0 & 0.915 & 2.31 & 4.34 & 6.74 & 9.05 & 10.93 & 12.22 & 12.92 & 13.1 & 12.99 \\
-21.19 & -21.16 & -20.82 & -19.94 & -18.38 & -16.04 & -12.84 & -9.07 & -5.42 & -2.42 & 0 & 2.42 & 5.42 & 9.07 & 12.84 & 16.04 & 18.38 & 19.94 & 20.82 & 21.16 & 21.19 \\
-30.16 & -30.09 & -29.75 & -28.91 & -27.4 & -25.04 & -21.59 & -16.6 & -10.82 & -5.48 & 0 & 5.48 & 10.82 & 16.6 & 21.59 & 25.04 & 27.4 & 28.91 & 29.75 & 30.09 & 30.16 \\
-39.62 & -39.67 & -39.67 & -39.29 & -38.21 & -36.28 & -33.46 & -29.67 & -23.25 & -12.73 & 0 & 12.73 & 23.25 & 29.67 & 33.46 & 36.28 & 38.21 & 39.29 & 39.67 & 39.67 & 39.62
\end{bmatrix}$$



Vertical joint displacements, $\cdot 10^{-3}$mm - $\mathbf{transp}(v)$ =

$$\begin{bmatrix} 9.79 & 17.93 & 34.35 & 62.36 & 79.37 & 93.17 & 104.42 & 113.38 & 119.96 & 124.01 & 125.37 & 124.01 & 119.96 & 113.38 & 104.42 & 93.17 & 79.37 & 62.36 & 34.35 & 17.93 & 9.79 \\ 14.45 & 26.77 & 42.83 & 62.86 & 80.38 & 94.24 & 105.67 & 114.8 & 121.52 & 125.65 & 127.04 & 125.65 & 121.52 & 114.8 & 105.67 & 94.24 & 80.38 & 62.86 & 42.83 & 26.77 & 14.45 \\ 21.47 & 34.19 & 48.85 & 65.24 & 81.15 & 95.02 & 106.6 & 115.93 & 122.83 & 127.09 & 128.53 & 127.09 & 122.83 & 115.93 & 106.6 & 95.02 & 81.15 & 65.24 & 48.85 & 34.19 & 21.47 \\ 28.77 & 39.98 & 53.29 & 67.81 & 82.34 & 95.77 & 107.44 & 117.01 & 124.18 & 128.63 & 130.14 & 128.63 & 124.18 & 117.01 & 107.44 & 95.77 & 82.34 & 67.81 & 53.29 & 39.98 & 28.77 \\ 34.78 & 44.59 & 56.7 & 70.03 & 83.63 & 96.63 & 108.34 & 118.22 & 125.77 & 130.53 & 132.16 & 130.53 & 125.77 & 118.22 & 108.34 & 96.63 & 83.63 & 70.03 & 56.7 & 44.59 & 34.78 \\ 39.13 & 48.09 & 59.31 & 71.81 & 84.77 & 97.5 & 109.34 & 119.67 & 127.79 & 133.01 & 134.82 & 133.01 & 127.79 & 119.67 & 109.34 & 97.5 & 84.77 & 71.81 & 59.31 & 48.09 & 39.13 \\ 41.88 & 50.54 & 61.2 & 73.1 & 85.61 & 98.2 & 110.33 & 121.34 & 130.33 & 136.31 & 138.41 & 136.31 & 130.33 & 121.34 & 110.33 & 98.2 & 85.61 & 73.1 & 61.2 & 50.54 & 41.88 \\ 43.3 & 52.09 & 62.43 & 73.89 & 86.06 & 98.58 & 111.11 & 123.07 & 133.41 & 140.59 & 143.18 & 140.59 & 133.41 & 123.07 & 111.11 & 98.58 & 86.06 & 73.89 & 62.43 & 52.09 & 43.3 \\ 43.79 & 52.91 & 63.09 & 74.23 & 86.11 & 98.52 & 111.37 & 124.52 & 136.88 & 146.02 & 149.4 & 146.02 & 136.88 & 124.52 & 111.37 & 98.52 & 86.11 & 74.23 & 63.09 & 52.91 & 43.79 \\ 43.8 & 53.23 & 63.32 & 74.24 & 85.88 & 98.1 & 110.9 & 125.06 & 140.48 & 152.68 & 157.05 & 152.68 & 140.48 & 125.06 & 110.9 & 98.1 & 85.88 & 74.24 & 63.32 & 53.23 & 43.8 \\ 43.75 & 53.3 & 63.33 & 74.11 & 85.56 & 97.58 & 110.24 & 124.05 & 143.81 & 160.41 & 164.51 & 160.41 & 143.81 & 124.05 & 110.24 & 97.58 & 85.56 & 74.11 & 63.33 & 53.3 & 43.75 \end{bmatrix}$$



Calculation of internal forces

Displacements for joint - $Z_j(j) = \mathbf{slice}(\vec{Z}; k_1 \cdot (j-1)+1; k_1 \cdot j)$

Displacements for element - $Z_e(e) = \left[Z_j\left(e_{j.e,1}\right); Z_j\left(e_{j.e,2}\right); Z_j\left(e_{j.e,3}\right); Z_j\left(e_{j.e,4}\right)\right]$

Membrane forces in element - $N_e(e; x; y) = -D \cdot B\left(\frac{2 \cdot x}{l_1}; \frac{2 \cdot y}{h_1}\right) \cdot Z_e(e)$

Results for element 101 and joint 111:

$\vec{Z}_e = Z_e(e) = Z_e(101) = [0\ \ 0.125\ \ 0\ \ 0.127\ \ \text{-}0.00919\ \ 0.124\ \ \text{-}0.00677\ \ 0.126]$ mm

$\vec{N}_e = N_e(e; x; y) = N_e(101; \text{-}0.1; \text{-}0.1) = [92.26\ \ 1.73\ \ 5.68]$ kN/m

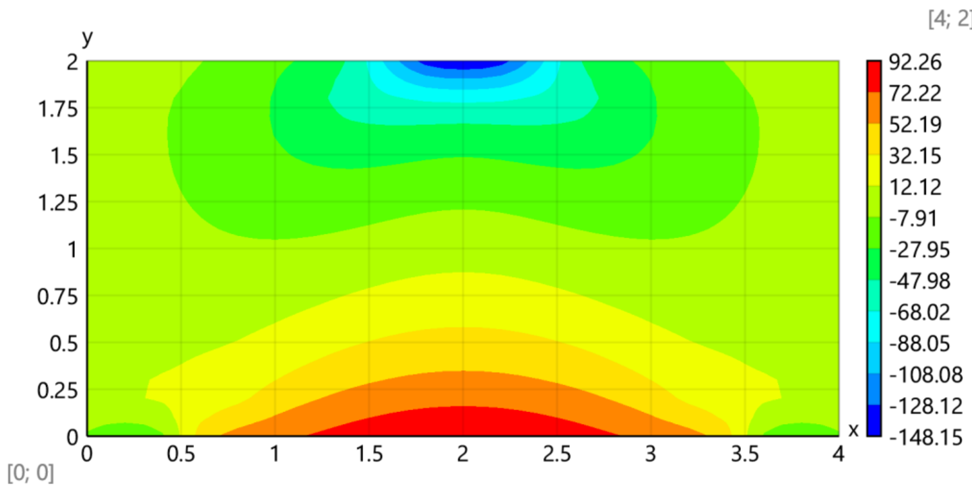
Average membrane forces at joints, kN/m - N_j =

$$\begin{bmatrix} -10.65 & 8.69 & 3.69 & -0.36 & -1.12 & -1.08 & -0.904 & -0.815 & -0.841 & -0.752 & 0.592 & -23.19 & 10.35 & 6.47 & 2.41 & 0.122 & -1.24 & -2.17 & -2.82 & -3.09 & \cdots & 0.592 \\ -48.71 & -56.68 & -70.86 & -66.62 & -52.02 & -35.69 & -21.04 & -9.71 & -2.66 & 0.0212 & 0.584 & -93.05 & -79.26 & -64.78 & -51.47 & -40.48 & -30.04 & -20.44 & -12.39 & -6.35 & \cdots & 0.584 \\ 13.55 & 0.204 & -2.13 & -5.1 & -6.23 & -6.04 & -5.13 & -3.79 & -2.23 & -0.9 & -0.376 & 5 & -3.67 & -7.63 & -11.24 & -12.49 & -12.01 & -10.32 & -7.8 & -4.84 & \cdots & 0.376 \end{bmatrix}$$

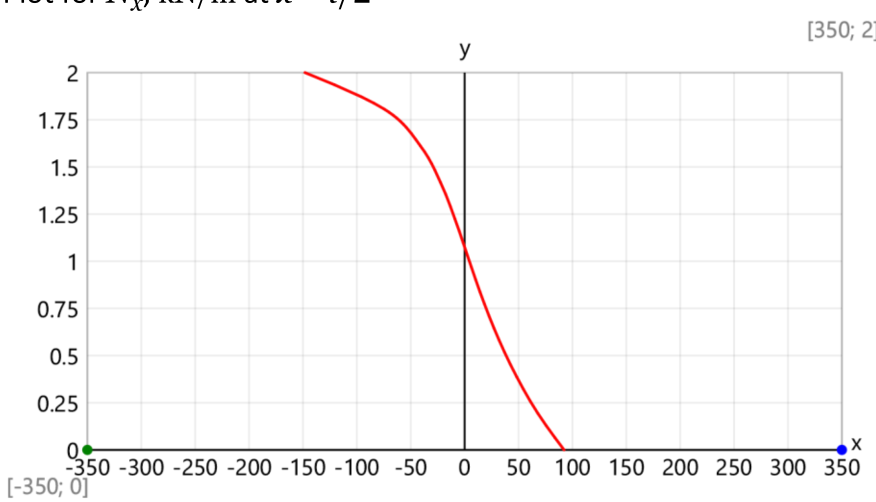
Membrane forces for the structure

Normal membrane forces - N_x, kN/m - $\mathbf{transp}(Nx)$ =

$$\begin{bmatrix} -10.65 & -23.19 & -10.26 & 39.56 & 59.56 & 66.82 & 73.73 & 80.76 & 86.78 & 90.83 & 92.26 & 90.83 & 86.78 & 80.76 & 73.73 & 66.82 & 59.56 & 39.56 & -10.26 & -23.19 & -10.65 \\ 8.69 & 10.35 & 14.14 & 20.22 & 27.99 & 41.47 & 50.65 & 57.64 & 62.83 & 66.09 & 67.21 & 66.09 & 62.83 & 57.64 & 50.65 & 41.47 & 27.99 & 20.22 & 14.14 & 10.35 & 8.69 \\ 3.69 & 6.47 & 9.96 & 13.3 & 17 & 23.81 & 31.81 & 38.32 & 43.14 & 46.14 & 47.16 & 46.14 & 43.14 & 38.32 & 31.81 & 23.81 & 17 & 13.3 & 9.96 & 6.47 & 3.69 \\ -0.36 & 2.41 & 4.2 & 6.06 & 8.39 & 12.22 & 17.34 & 22.56 & 26.82 & 29.61 & 30.58 & 29.61 & 26.82 & 22.56 & 17.34 & 12.22 & 8.39 & 6.06 & 4.2 & 2.41 & -0.36 \\ -1.12 & 0.122 & 0.080 & 0.187 & 0.844 & 2.65 & 5.72 & 9.5 & 13.12 & 15.71 & 16.65 & 15.71 & 13.12 & 9.5 & 5.72 & 2.65 & 0.844 & 0.187 & 0.080 & 0.122 & -1.12 \\ -1.08 & -1.24 & -2.79 & -4.56 & -5.84 & -6 & -4.7 & -2.14 & 0.93 & 3.43 & 4.39 & 3.43 & 0.93 & -2.14 & -4.7 & -6 & -5.84 & -4.56 & -2.79 & -1.24 & -1.08 \\ -0.904 & -2.17 & -4.96 & -8.51 & -11.89 & -14.27 & -14.96 & -13.7 & -11.11 & -8.56 & -7.49 & -8.56 & -11.11 & -13.7 & -14.96 & -14.27 & -11.89 & -8.51 & -4.96 & -2.17 & -0.904 \\ -0.815 & -2.82 & -6.55 & -11.57 & -17.05 & -22.13 & -25.57 & -26.37 & -24.69 & -22.16 & -20.98 & -22.16 & -24.69 & -26.37 & -25.57 & -22.13 & -17.05 & -11.57 & -6.55 & -2.82 & -0.815 \\ -0.841 & -3.09 & -7.25 & -13.08 & -20.15 & -28.34 & -36.09 & -40.74 & -42.01 & -40.81 & -39.61 & -40.81 & -42.01 & -40.74 & -36.09 & -28.34 & -20.15 & -13.08 & -7.25 & -3.09 & -0.841 \\ -0.752 & -2.58 & -6.36 & -12.07 & -19.61 & -29.3 & -42.76 & -55.56 & -65.15 & -71.38 & -72.8 & -71.38 & -65.15 & -55.56 & -42.76 & -29.3 & -19.61 & -12.07 & -6.36 & -2.58 & -0.752 \\ 0.592 & 0.122 & -2 & -7.34 & -15.02 & -23.66 & -33.05 & -51.1 & -95.19 & -137.19 & -148.15 & -137.19 & -95.19 & -51.1 & -33.05 & -23.66 & -15.02 & -7.34 & -2 & 0.122 & 0.592 \end{bmatrix}$$



Plot for N_x, kN/m at $x = l/2$

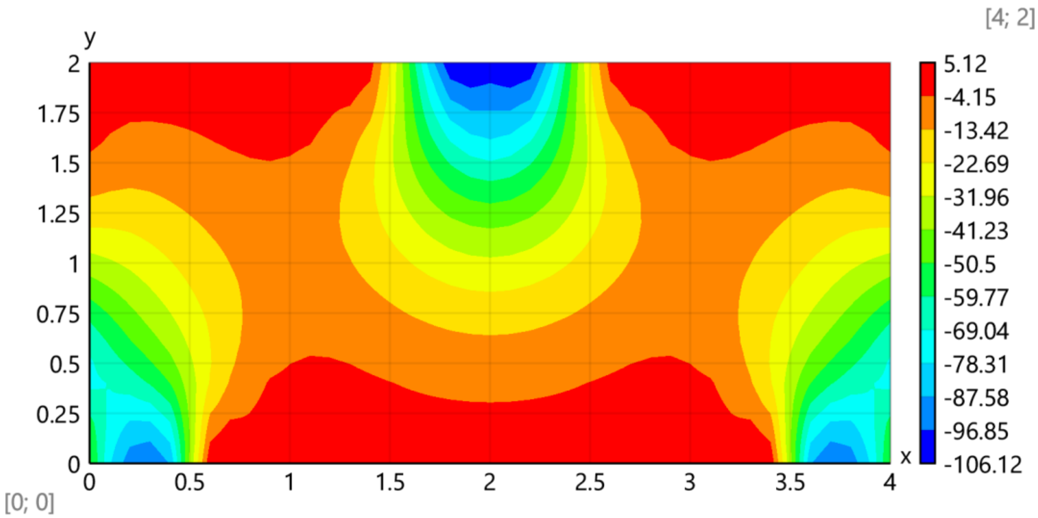


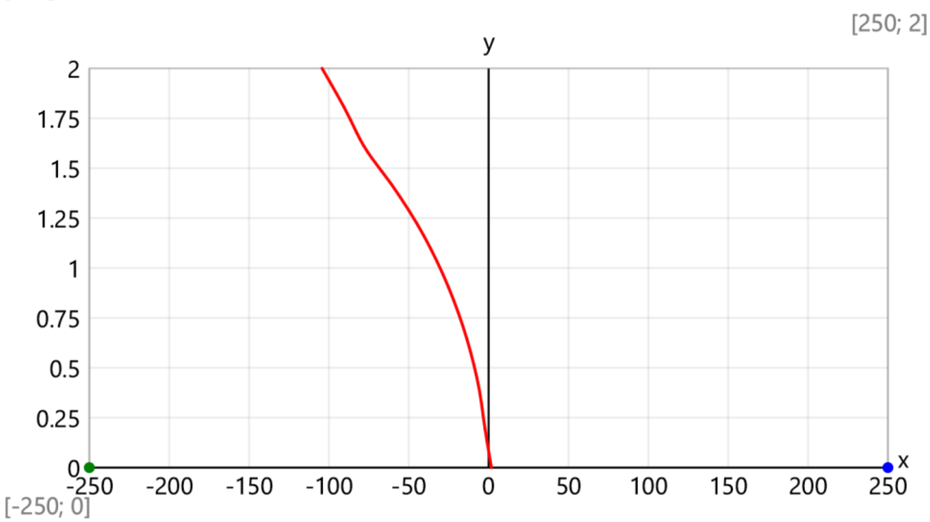
Bottom value - $N_x\left(\dfrac{l}{2};0\right) = N_x\left(\dfrac{4}{2};0\right) = 92.26\ \mathrm{kN/m}$

Top value - $N_x\left(\dfrac{l}{2};h\right) = N_x\left(\dfrac{4}{2};2\right) = -148.15\ \mathrm{kN/m}$

Normal membrane forces - N_y, kN/m - $\mathbf{transp}(Ny)$ =

$$\begin{bmatrix}
-48.71 & -93.05 & -86.8 & 2.92 & 1.76 & 2.7 & 2.2 & 2 & 1.85 & 1.76 & 1.73 & 1.76 & 1.85 & 2 & 2.2 & 2.7 & 1.76 & 2.92 & -86.8 & -93.05 & -48.71 \\
-56.68 & -79.26 & -69.65 & -10.32 & -3.3 & -0.933 & -0.794 & -1.23 & -1.78 & -2.2 & -2.36 & -2.2 & -1.78 & -1.23 & -0.794 & -0.933 & -3.3 & -10.32 & -69.65 & -79.26 & -56.68 \\
-70.86 & -64.78 & -50.3 & -22.09 & -6.41 & -2.88 & -2.47 & -3.41 & -4.68 & -5.69 & -6.07 & -5.69 & -4.68 & -3.41 & -2.47 & -2.88 & -6.41 & -22.09 & -50.3 & -64.78 & -70.86 \\
-66.62 & -51.47 & -38.41 & -22.78 & -10.75 & -5.63 & -5.24 & -6.94 & -9.32 & -11.26 & -12.01 & -11.26 & -9.32 & -6.94 & -5.24 & -5.63 & -10.75 & -22.78 & -38.41 & -51.47 & -66.62 \\
-52.02 & -40.48 & -30.1 & -19.93 & -11.95 & -8.11 & -8.38 & -11.4 & -15.44 & -18.78 & -20.08 & -18.78 & -15.44 & -11.4 & -8.38 & -8.11 & -11.95 & -19.93 & -30.1 & -40.48 & -52.02 \\
-35.69 & -30.04 & -23.07 & -16.21 & -11.05 & -9.07 & -10.89 & -15.99 & -22.62 & -28.21 & -30.39 & -28.21 & -22.62 & -15.99 & -10.89 & -9.07 & -11.05 & -16.21 & -23.07 & -30.04 & -35.69 \\
-21.04 & -20.44 & -16.6 & -12.13 & -8.84 & -8.3 & -11.8 & -19.72 & -30.32 & -39.58 & -43.26 & -39.58 & -30.32 & -19.72 & -11.8 & -8.3 & -8.84 & -12.13 & -16.6 & -20.44 & -21.04 \\
-9.71 & -12.39 & -10.77 & -8 & -5.93 & -5.98 & -10.27 & -21.2 & -37.7 & -52.98 & -59.15 & -52.98 & -37.7 & -21.2 & -10.27 & -5.98 & -5.93 & -8 & -10.77 & -12.39 & -9.71 \\
-2.66 & -6.35 & -5.89 & -4.35 & -3.11 & -3.23 & -6.19 & -18.09 & -43.78 & -68.64 & -77.28 & -68.64 & -43.78 & -18.09 & -6.19 & -3.23 & -3.11 & -4.35 & -5.89 & -6.35 & -2.66 \\
0.0212 & -2.5 & -2.47 & -1.77 & -1.17 & -1.16 & -2.94 & -8.75 & -47.68 & -86.25 & -90.13 & -86.25 & -47.68 & -8.75 & -2.94 & -1.16 & -1.17 & -1.77 & -2.47 & -2.5 & 0.0212 \\
0.584 & -0.693 & -0.515 & -0.141 & 0.139 & 0.461 & -0.0283 & -0.167 & -52.35 & -104.7 & -104.3 & -104.7 & -52.35 & -0.167 & -0.0283 & 0.461 & 0.139 & -0.141 & -0.515 & -0.693 & 0.584
\end{bmatrix}$$

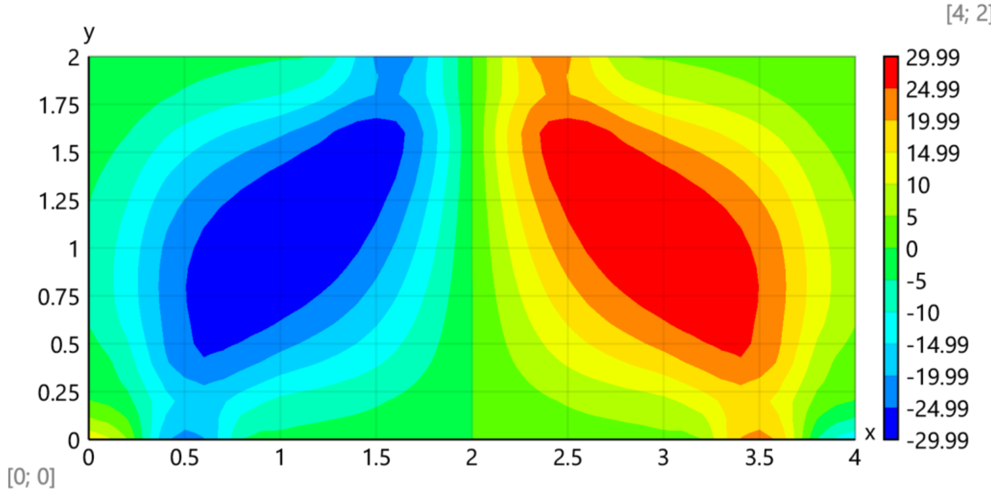




Top value - $N_y\left(\dfrac{l}{2};h\right) = N_y\left(\dfrac{4}{2};2\right) = -104.3\ \mathrm{kN/m}$

Shear membrane forces - N_{xy}, kN/m - $\mathbf{transp}(Nxy)$ =

$$\begin{bmatrix} 13.55 & 5 & -19.47 & -20.12 & -4.98 & -4.01 & -3.42 & -2.94 & -2.18 & -1.17 & 0 & 1.17 & 2.18 & 2.94 & 3.42 & 4.01 & 4.98 & 20.12 & 19.47 & -5 & -13.55 \\ 0.204 & -3.67 & -13.65 & -17.4 & -12.68 & -8.91 & -7.6 & -6.3 & -4.6 & -2.44 & 0 & 2.44 & 4.6 & 6.3 & 7.6 & 8.91 & 12.68 & 17.4 & 13.65 & 3.67 & -0.204 \\ -2.13 & -7.63 & -19.42 & -24.62 & -21.66 & -17.92 & -14.99 & -12.21 & -8.8 & -4.64 & 0 & 4.64 & 8.8 & 12.21 & 14.99 & 17.92 & 21.66 & 24.62 & 19.42 & 7.63 & 2.13 \\ -5.1 & -11.24 & -21.03 & -26.5 & -26.93 & -24.52 & -21.33 & -17.48 & -12.62 & -6.66 & 0 & 6.66 & 12.62 & 17.48 & 21.33 & 24.52 & 26.93 & 26.5 & 21.03 & 11.24 & 5.1 \\ -6.23 & -12.49 & -21.72 & -27.13 & -29.07 & -28.4 & -25.95 & -21.93 & -16.14 & -8.61 & 0 & 8.61 & 16.14 & 21.93 & 25.95 & 28.4 & 29.07 & 27.13 & 21.72 & 12.49 & 6.23 \\ -6.04 & -12.01 & -20.76 & -26.22 & -29.13 & -29.95 & -28.85 & -25.51 & -19.42 & -10.6 & 0 & 10.6 & 19.42 & 25.51 & 28.85 & 29.95 & 29.13 & 26.22 & 20.76 & 12.01 & 6.04 \\ -5.13 & -10.32 & -18.14 & -23.5 & -27.07 & -29.26 & -29.93 & -28.15 & -22.59 & -12.74 & 0 & 12.74 & 22.59 & 28.15 & 29.93 & 29.26 & 27.07 & 23.5 & 18.14 & 10.32 & 5.13 \\ -3.79 & -7.8 & -14.17 & -19.01 & -22.74 & -25.88 & -28.59 & -29.43 & -25.49 & -15.08 & 0 & 15.08 & 25.49 & 29.43 & 28.59 & 25.88 & 22.74 & 19.01 & 14.17 & 7.8 & 3.79 \\ -2.23 & -4.84 & -9.37 & -13.18 & -16.26 & -19.3 & -23.25 & -27.63 & -27.04 & -16.57 & 0 & 16.57 & 27.04 & 27.63 & 23.25 & 19.3 & 16.26 & 13.18 & 9.37 & 4.84 & 2.23 \\ -0.9 & -2.1 & -4.48 & -6.68 & -8.38 & -9.95 & -13.2 & -18.7 & -20.41 & -13.02 & 0 & 13.02 & 20.41 & 18.7 & 13.2 & 9.95 & 8.38 & 6.68 & 4.48 & 2.1 & 0.9 \\ -0.376 & -0.852 & -1.96 & -3.06 & -3.85 & -4.56 & -5.67 & -15.47 & -23.98 & -12.91 & 0 & 12.91 & 23.98 & 15.47 & 5.67 & 4.56 & 3.85 & 3.06 & 1.96 & 0.852 & 0.376 \end{bmatrix}$$

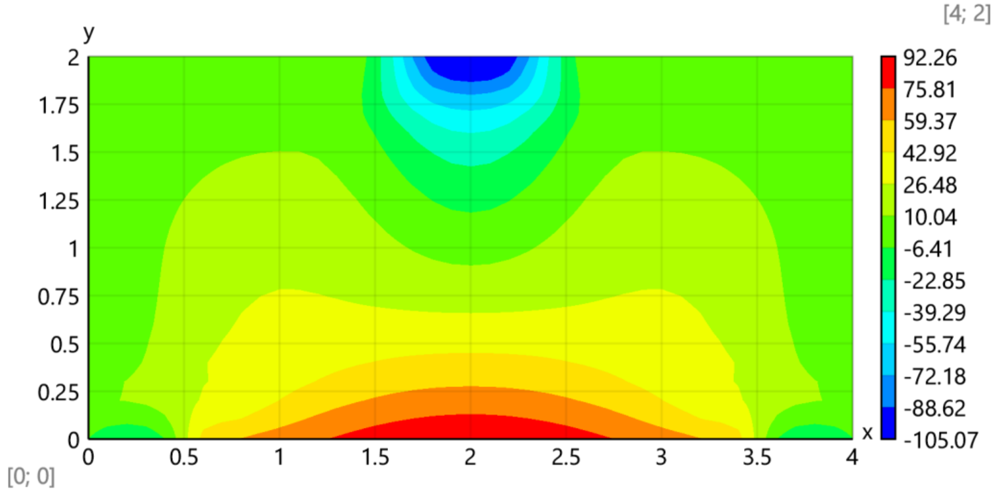


Max. value at 3/4 of span - $N_{xy}\left(\frac{3 \cdot l}{4};\frac{h}{2}\right) = N_{xy}\left(\frac{3 \cdot 4}{4};\frac{2}{2}\right) = 29.95\ \text{kN/m}$

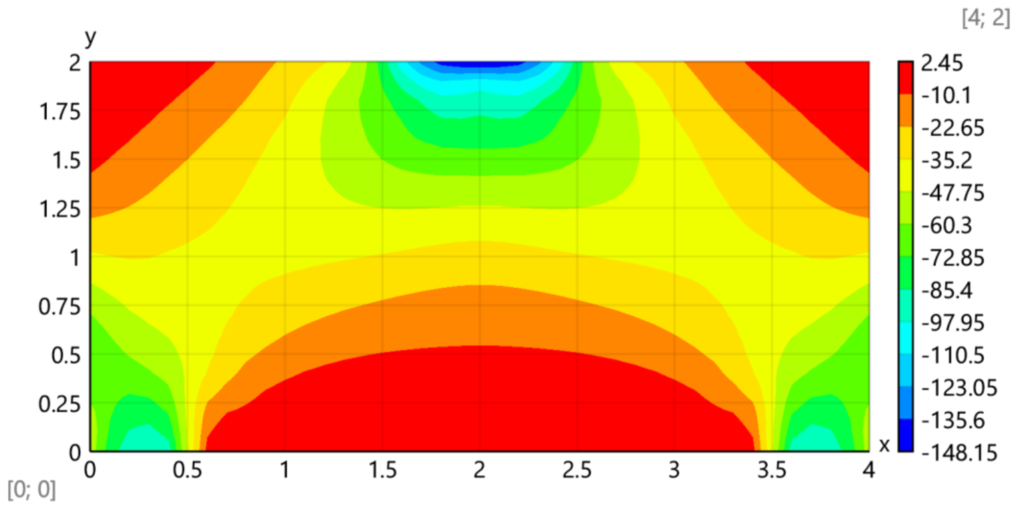
Principal membrane forces, kN/m

$N_m(x;y) = 0.5 \cdot \left(N_x(x;y) + N_y(x;y)\right), \quad \Delta N(x;y) = 0.5 \cdot \left(\sqrt{\left(N_x(x;y) - N_y(x;y)\right)^2 + 4 \cdot N_{xy}(x;y)^2}\right)$

$N_{max}(x;y) = N_m(x;y) + \Delta N(x;y)$ kN/m

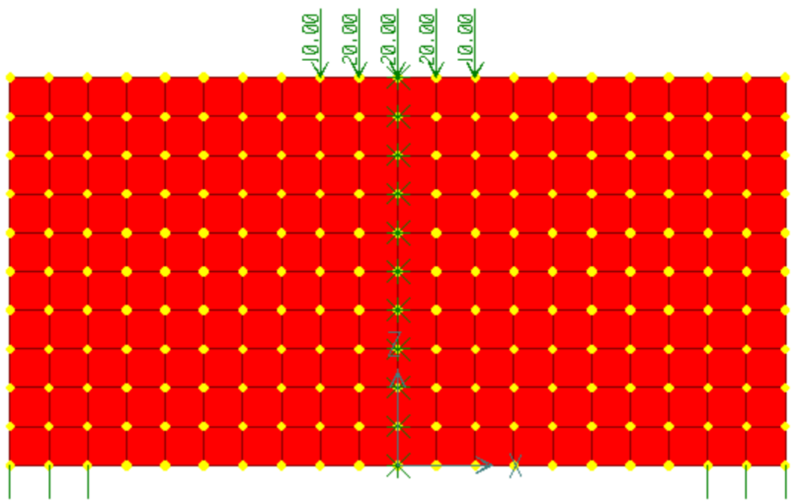


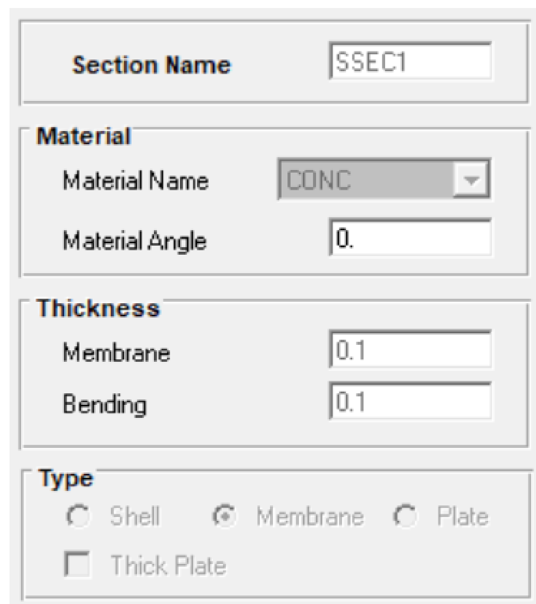
$N_{min}(x;y) = N_m(x;y) - \Delta N(x;y)$ kN/m

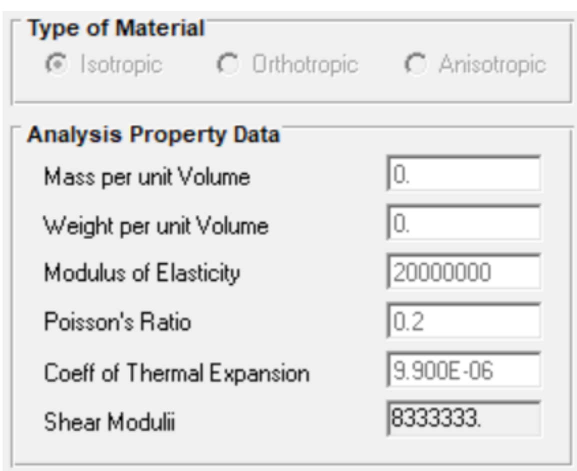


Solution with SAP 2000

Input data

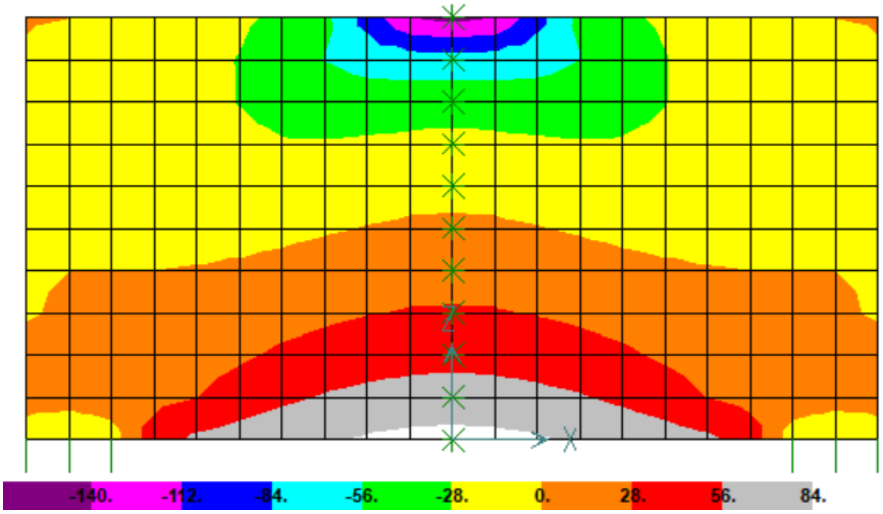


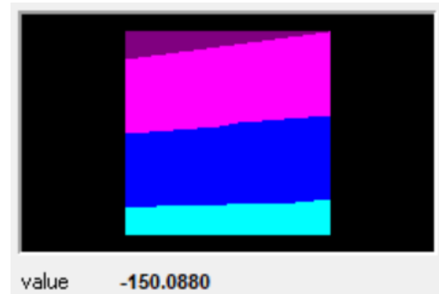




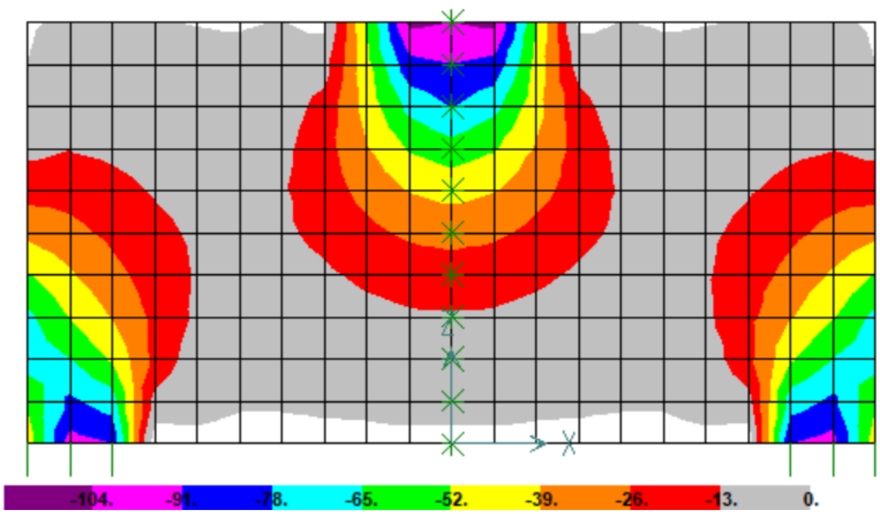
Results

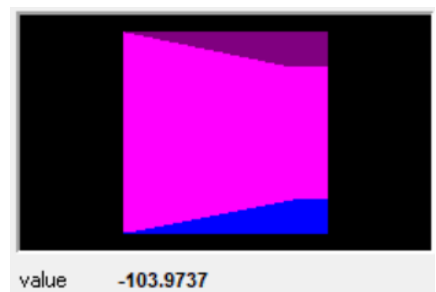
Normal membrane forces - N_x, kN/m (F11)

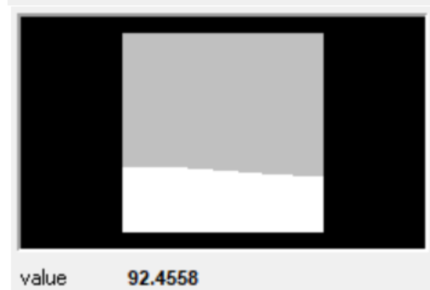




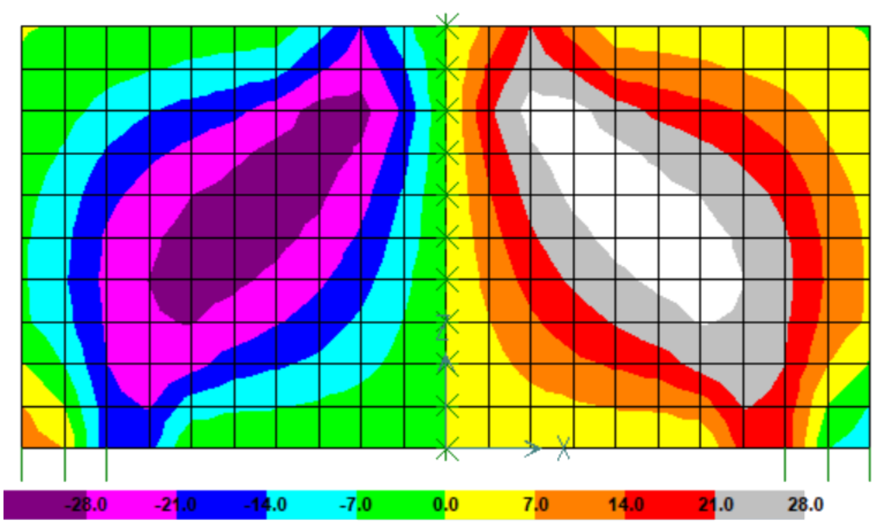
Normal membrane forces - N_y, kN/m (F22)



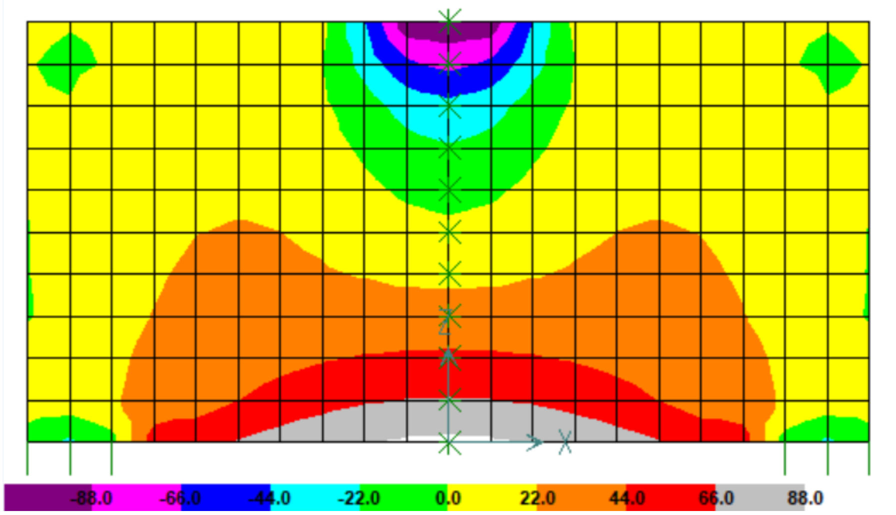




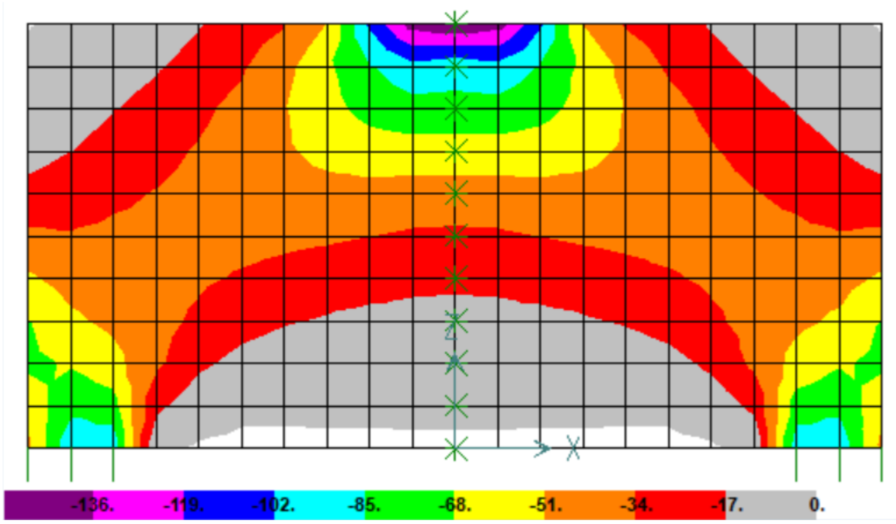
Shear membrane forces - N_{xy}, kN/m (F12)



Principal membrane forces - N_{max}, kN/m (FMAX)



Principal membrane forces - N_{min}, kN/m (FMIN)

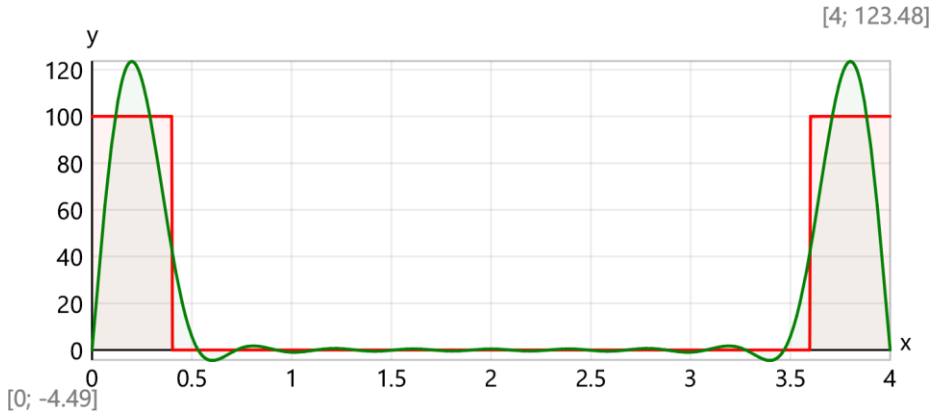


Analytical solution – Fourier series

Number of iterations - $N = 21$

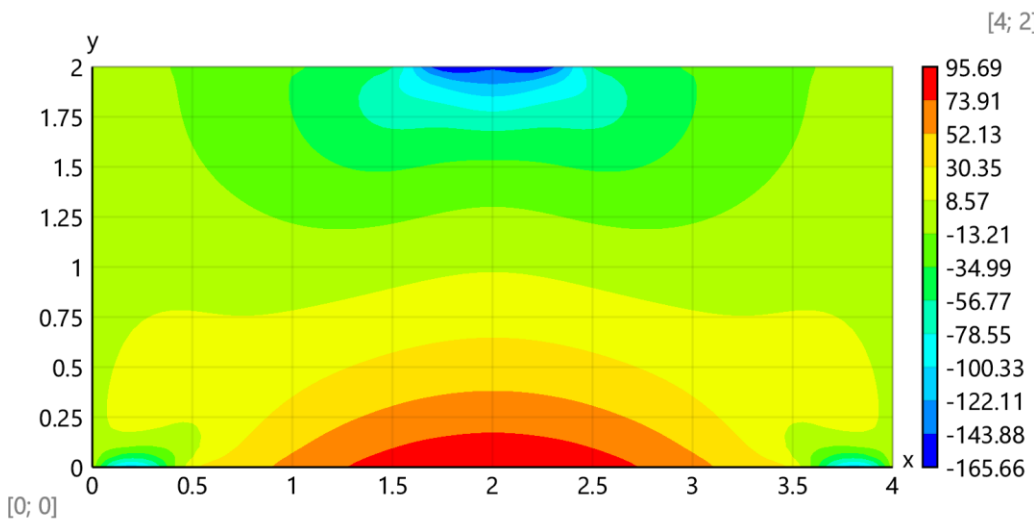
Original and Fourier functions for the load

Original and Fourier functions for the reaction in supports

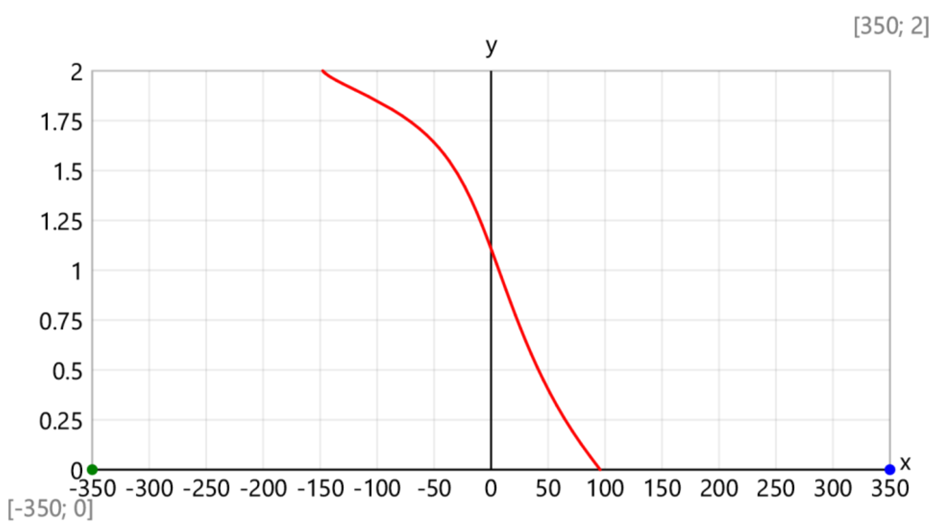


Calculation of stresses

$$\sigma_x(x;y) = \sum_{n=1}^{N} Y_2(n;y) \cdot \sin(\alpha(n) \cdot x)$$



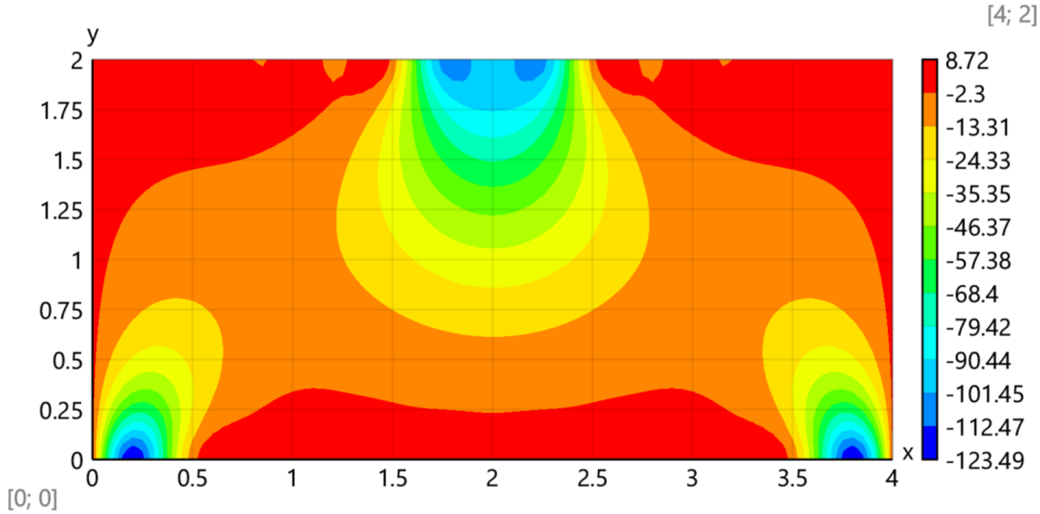
Plot for σ_x at $x = l/2$



Bottom value - $\sigma_x\left(\frac{l}{2}; 0\,\text{m}\right) = \sigma_x\left(\frac{4\,\text{m}}{2}; 0\,\text{m}\right) = 95.66\,\text{kN/m}$

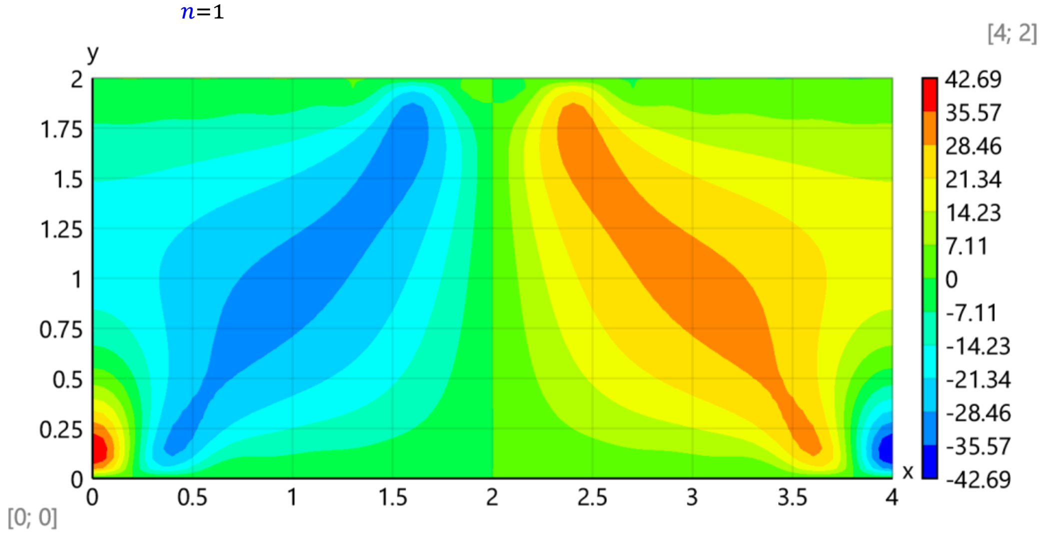
Top value - $\sigma_x\left(\frac{l}{2}; h\right) = \sigma_x\left(\frac{4\,\text{m}}{2}; 2\,\text{m}\right) = \text{-}147.56\,\text{kN/m}$

$$\sigma_y(x; y) = -\sum_{n=1}^{N} \alpha(n)^2 \cdot Y(n; y) \cdot \sin(\alpha(n) \cdot x)$$



Middle top value - $\sigma_y\left(\frac{l}{2}; h\right) = \sigma_y\left(\frac{4\,\text{m}}{2}; 2\,\text{m}\right) = \text{-}92.32\,\text{kN/m}$

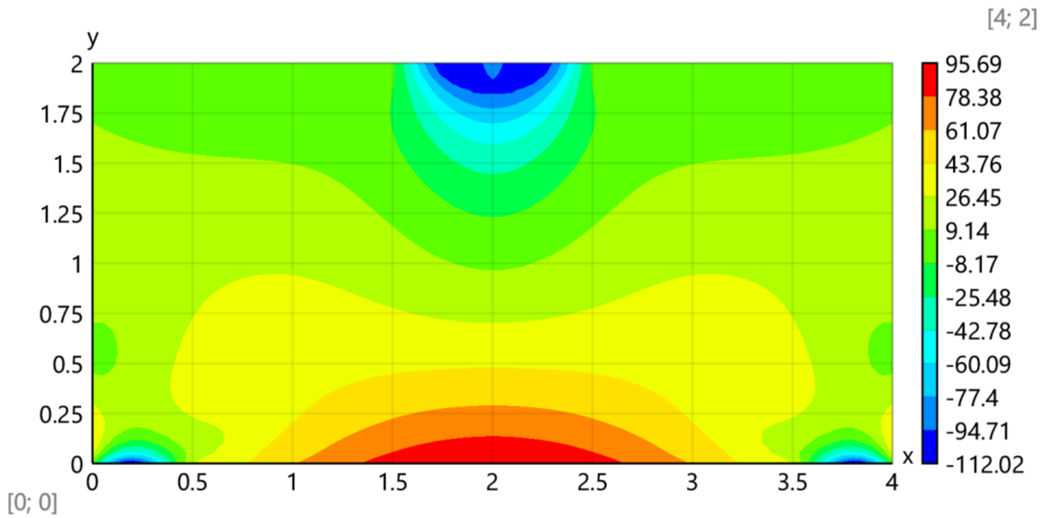
$$\tau_{xy}(x; y) = -\sum_{n=1}^{N} \alpha(n) \cdot Y_1(n; y) \cdot \cos(\alpha(n) \cdot x)$$



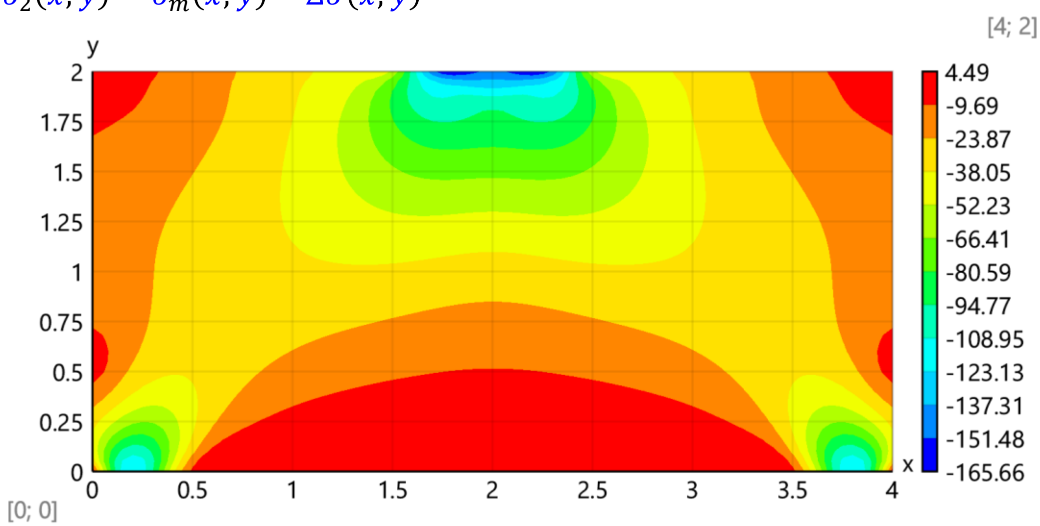
Value at 3/4 of span - $\tau_{xy}\left(\frac{3 \cdot l}{4}; \frac{h}{2}\right) = \tau_{xy}\left(\frac{3 \cdot 4\,\text{m}}{4}; \frac{2\,\text{m}}{2}\right) = 29.96\,\text{kN/m}$

$$\sigma_m(x; y) = 0.5 \cdot \left(\sigma_x(x; y) + \sigma_y(x; y)\right), \quad \Delta\sigma(x; y) = 0.5 \cdot \sqrt{\left(\sigma_x(x; y) - \sigma_y(x; y)\right)^2 + 4 \cdot \tau_{xy}(x; y)^2}$$

$$\sigma_1(x; y) = \sigma_m(x; y) + \Delta\sigma(x; y)$$



$\sigma_2(x; y) = \sigma_m(x; y) - \Delta\sigma(x; y)$



Comparison of the results

	Analytical	FEA Calcpad	FEA SAP 2000
$N_{x,\mathrm{btm}}$, kN/m	95,66	92,26	92,46
$N_{x,\mathrm{top}}$, kN/m	-147,56	-148,15	-150
N_{y}, kN/m	-92,32	-104,3	-103,97
N_{xy}, kN/m	29,96	29,95	30,91

Difference, %

	Analytical	FEA Calcpad	FEA SAP 2000
$N_{x,\mathrm{btm}}$, kN/m	0,00%	-3,55%	-3,35%
$N_{x,\mathrm{top}}$, kN/m	0,00%	0,40%	1,65%
N_{y}, kN/m	0,00%	12,98%	12,62%
N_{xy}, kN/m	0,00%	-0,03%	3,17%