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Stability Analysis of Social Foraging Swarms: Combined Effects of Attractant/Repellent Profiles

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Abstract

In this article we study the stability of the collective behavior of social foraging swarms, i.e., swarms moving in a profile of nutrient/toxic substances (an attractant/repellent profile) and extend our results in [1]. In particular, we consider a plane profile and also extend our results for the quadratic, Gaussian, and multi-modal Gaussian profiles. Moreover, we analyze the ultimate behavior of the individuals in the social foraging swarm. The paper closes with new simulation studies that give insights into swarm dynamics.

1 Introduction

Swarming, or aggregations of organisms in groups, can be found in nature in many organisms ranging from simple bacteria to mammals. Such behavior can result from several different mechanisms. For example, individuals may respond directly to local physical cues such as concentration of nutrients or distribution of some chemicals (which may be laid by other individuals). This process is called *chemotaxis* and is used by organisms such as bacteria or social insects (e.g., by ants in trail following or by honey bees in cluster formation). As another example, individuals may respond directly to other individuals (rather than the cues they leave about their activities) as seen in some higher organisms such as fish, birds, and herds of mammals.

Evolution of swarming behavior is driven by the advantages of such collective and coordinated behavior for avoiding predators and increasing the chance of finding food. For example, in [2, 3] Grünbaum explains how social foragers as a group more successfully perform chemotaxis over noisy gradients than individually. In other words, individuals do much better collectively compared to the case when they forage on their own. Operational principles from such biological systems can be used in engineering for developing distributed cooperative control, coordination, and learning strategies for autonomous multi-agent systems such as autonomous multi-robot applications, unmanned undersea, land, or air vehicles. The development of such highly automated systems is likely to benefit from biological principles including modeling of biological swarms, coordination strategy specification, and analysis to show that group dynamics achieve group goals. In [1] we developed a simple M-member "individual-based" continuous time model of swarming in the presence of an attractant/repellent or nutrient profile. In our model the motion of each individual is determined by three factors: (i) attraction to the other individuals on long distances, (ii) repulsion from the other individuals on short distances, (iii) attraction to the more favorable regions (or repulsion from the unfavorable regions) of the attractant/repellent profile. There we analyzed the stability properties of the model for different profiles. However, there are some cases that were not considered there. For example, there we did not consider a plane profile and considered only valleys of the quadratic and the Gaussian profiles. Here we consider the plane profile as well as profiles that consist of hills, and also extend the results for the multimodal Gaussian profiles. Furthermore, we analyze the ultimate behavior of the individuals in the swarm, which was not done in [1]. We illustrate the theory with simulation examples.

2 The Swarm Model

As in [1, 4, 5] we consider a swarm of M individuals (members) in an n-dimensional Euclidean space. We model the individuals as points and ignore their dimensions. The position of member i of the swarm is described by $x^i \in \mathbb{R}^n$. We assume synchronous motion and no time delays, i.e., all the individuals move simultaneously and know the exact position of all the other individuals. Let $\sigma : \mathbb{R}^n \to \mathbb{R}$ represent the attractant/repellent profile or the "σ-profile" which can be a profile of nutrients or some attractant or repellent substances (e.g., food/nutrients, pheromones laid by other individual, or toxic chemicals). Assume that the areas that are minimum points are favorable by the individuals in the swarm. For example, assume that $\sigma(y) < 0$ represents attractant or nutrient rich, $\sigma(y) = 0$ represents a neutral, and $\sigma(y) > 0$ represents a noxious environment at y.

We consider the equation of motion of each individual i [1]

$$\dot{x}^i = -\nabla_{x^i}\sigma(x^i) + \sum_{j=1, j\neq i}^{M} g(x^i - x^j), i = 1, \ldots, M, \tag{1}$$

where $g(\cdot)$ represents the function of mutual attraction and repulsion between the individuals. In this article we will consider the function considered in [4] which is a special case of the functions considered in [5] and is given by

$$g(y) = -y\left(a - b\exp\left(-\frac{\|y\|^2}{c}\right)\right), \tag{2}$$

where a, b, and c are positive constants such that $b > a$, and $\|y\|$ is the Euclidean norm $\|y\| = \sqrt{y^\top y}$. Defining the *center* of the swarm as $\bar{x} = \frac{1}{M}\sum_{i=1}^{M} x^i$ it was shown in [1] that

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$$\dot{\bar{x}} = -\frac{1}{M}\sum_{i=1}^{M} \nabla_{x^i}\sigma(x^i). \quad (3)$$

Remark: Note that the collective behavior in Eq. (3) has a kind of averaging (filtering or smoothing) effect. This may be important if the underlying profile function is a noisy function (or there is a measurement error or noise in the system as discussed in [2, 3]). ■

3 Motion Along a Plane Attractant/Repellent Profile

Assume that the profile is described by

$$\sigma(y) = a_\sigma^\top y + b_\sigma, \quad (4)$$

where $a_\sigma \in \mathbb{R}^n$ and $b_\sigma \in \mathbb{R}$. Then, we have $\nabla_y \sigma(y) = a_\sigma$. and $\dot{\bar{x}} = -\frac{1}{M}\sum_{i=1}^{M} a_\sigma = -a_\sigma$. This equation implies that the center of the swarm will be moving with the constant velocity vector $-a_\sigma$ (and eventually will diverge towards infinity where the minimum of the profile occurs). Note that the above motion can be viewed as a model of a foraging herd that moves in a constant direction with a constant speed such as the one considered in [6] or it can be viewed as a model of group of autonomous agents moving in a formation with a constant speed. The only drawback of this view is that we cannot a priori specify the formation to be established.

Next, we need to analyze the cohesiveness of the swarm during its journey to infinity, which is done in the following result.

Theorem 1 *Consider the swarm described by the model in Eq. (1) with an attraction/repulsion function $g(\cdot)$ as given in Eq. (2). Assume that the σ-profile of the environment is given by Eq. (4). Then, in a finite time bounded by*

$$\bar{t} = \max_{i\in S}\left\{-\frac{1}{2a}\ln\left(\frac{\varepsilon_1^2}{2V_i(0)}\right)\right\},$$

where
$$\varepsilon_1 = \frac{b}{a}\sqrt{\frac{c}{2}}\exp\left(-\frac{1}{2}\right).$$

all the members of the swarm will converge to (and will stay within for all $t \geq \bar{t}$) the hyperball

$$B_{\varepsilon_1}(\bar{x}) = \{y : \|y - \bar{x}\| \leq \varepsilon_1\}.$$

Proof: Similar to the proof Theorem 1 in [4]. ■

4 Quadratic Attractant/Repellent Profiles

Consider another simple profile,

$$\sigma(y) = \frac{A_\sigma}{2}\|y - c_\sigma\|^2 + b_\sigma, \quad (5)$$

where $A_\sigma \in \mathbb{R}$, $b_\sigma \in \mathbb{R}$, and $c_\sigma \in \mathbb{R}^n$. Note that this profile has a global extremum (either a minimum or a maximum depending on the sign of A_σ) at $y = c_\sigma$. Defining the error between the center $\bar{x}$ and the extremum point c_σ as $e_\sigma = \bar{x} - c_\sigma$ it can be shown that [1] $\dot{e}_\sigma = \dot{\bar{x}} = -A_\sigma e_\sigma$. Note that in [1] we considered the case in which $A_\sigma > 0$ and concluded that as $t \to \infty$ we have $e_\sigma \to 0$. If $A_\sigma < 0$ we will have two different cases.

Case 1 $\bar{x}(0) \neq c_\sigma$: For this case from the above error equation we conclude that as $t \to \infty$ we have $\bar{x} \to \infty$ (i.e., the center of the swarm diverges from the global maximum c_σ of the profile). In other words, for any $D > 0$ (no matter how large) it can be shown that $\|\bar{x} - c_\sigma\| > D$ is satisfied in a finite time, implying that $\|\bar{x}\|$ leaves any bounded D-neighborhood of c_σ in a finite time.

Case 2 $\bar{x}(0) = c_\sigma$: If this is the case then we will have $\bar{x} = c_\sigma$ for all t. In other words, for this case the swarm will be either "trapped" around the maximum point because of the interindividual attraction (i.e., desire of the individuals to be close to each other) or will disperse in all directions if the interindividual attraction is not strong enough. Note, however, that even if they disperse the center $\bar{x}$ will not move and stay at c_σ.

Lemma 1 *Consider the swarm described by the model in Eq. (1) with interindividual attraction/repulsion function $g(\cdot)$ as given in Eq. (2). Assume that the σ-profile of the environment is given by Eq. (5). As $t \to \infty$ we have*

- *If $A_\sigma > 0$, then $\bar{x} \to c_\sigma$ [1].*
- *If $A_\sigma < 0$ and $\bar{x}(0) \neq c_\sigma$, then $\bar{x} \to \infty$.*

The above analysis concerns the motion of the center of the swarm. However, it does not imply anything about the cohesiveness of the swarm. Considering the cohesiveness of the swarm, we see that Lemma 2 in [1] holds also for the case $A_\sigma < 0$ provided that $A_\sigma > -aM$. In other words, the statement of the lemma can be modified as follows.

Lemma 2 *Consider the swarm described by the model in Eq. (1) with interindividual attraction/repulsion function $g(\cdot)$ as given in Eq. (2). Assume that the σ-profile of the environment is given by Eq. (5) and that $A_\sigma > -aM$. Then, as $t \to \infty$ for all individuals $i = 1,\ldots,M$, we have $x^i \to B_{\varepsilon_2}(\bar{x})$ (i.e., all individuals converge to and stay within the hyperball $B_{\varepsilon_2}(\bar{x})$), where*

$$\varepsilon_2 = \frac{b(M-1)}{aM + A_\sigma}\sqrt{\frac{c}{2}}\exp\left(-\frac{1}{2}\right).$$

The above result implies that as $t \to \infty$, asymptotically we will have $\|e^i\| \leq \varepsilon_2$. Note, however, that for any small $\lambda > 0$ and $\varepsilon > \varepsilon_2$ defined as

$$\varepsilon = \frac{b(M-1)}{aM + A_\sigma - \lambda}\sqrt{\frac{c}{2}}\exp\left(-\frac{1}{2}\right)$$

one can show that for all $i = 1,\ldots,M$, the individual position x^i will enter $B_\varepsilon(\bar{x})$ in a finite time. This observation, together with Lemmas 1 and 2 lead us to the following result.

Theorem 2 *Consider the swarm described by the model in Eq. (1) with interindividual attraction/repulsion function $g(\cdot)$ as given in Eq. (2). Assume that the σ-profile of the environment is given by Eq. (5) and that $A_\sigma > -aM$. Then, the following hold*

- *If $A_\sigma > 0$, then for any $\varepsilon > \varepsilon_2$ all individuals $i = 1,\ldots,M$, will enter $B_\varepsilon(c_\sigma)$ in a finite time [1],*

- *If $A_\sigma < 0$ and and $\bar{x}(0) \neq c_\sigma$, then for any $D < \infty$ all individuals $i = 1,\ldots,M$, will exit $B_D(c_\sigma)$ in a finite time.*

5 Gaussian Attractant/Repellent Profiles

Consider a profile given by

$$\sigma(y) = -\frac{A_\sigma}{2}\exp\left(-\frac{\|y-c_\sigma\|^2}{l_\sigma}\right) + b_\sigma, \qquad (6)$$

where $A_\sigma \in \mathbb{R}$, $b_\sigma \in \mathbb{R}$, $l_\sigma \in \mathbb{R}^+$, and $c_\sigma \in \mathbb{R}^n$.

For this case one can show that Lemma 3 in [1] holds (with only a small modification due to the fact that A_σ can be negative). We repeat it here for conveniance of the reader.

Lemma 3 *[1] Consider the swarm described by the model in Eq. (1) with interindividual attraction/repulsion function $g(\cdot)$ as given in Eq. (2). Assume that the σ-profile of the environment is given by Eq. (6). Then, as $t \to \infty$ for all individuals $i = 1,\ldots,M$ we have $x^i \to B_{\varepsilon_3}(\bar{x})$, for*

$$\varepsilon_3 = \frac{b(M-1)}{aM}\sqrt{\frac{c}{2}}\exp\left(-\frac{1}{2}\right) + \frac{|A_\sigma|}{aM}\sqrt{\frac{2}{l_\sigma}}\exp\left(-\frac{1}{2}\right).$$

This result shows that as time progresses the individuals will form a cohesive swarm around the center $\bar{x}$ and will preserve its cohesiveness during motion. Now, we have to analyze the motion of $\bar{x}$ in order to determine the overall behavior of the swarm.

Lemma 4 *Consider the swarm described by the model in Eq. (1) with interindividual attraction/repulsion function $g(\cdot)$ as given in Eq. (2). Assume that the σ-profile of the environment is given by Eq. (6). Then, as $t \to \infty$ we have*

- *If $A_\sigma > 0$, then $\|e_\sigma\| \le e_{max} = \max_{i=1,\ldots,M}\|e^i\|$, [1]*

- *If $A_\sigma < 0$ and $\|e_\sigma(0)\| > e_{max}(0)$ (here we assume that $x^i(0) \neq x^j(0)$ for at least one pair of individuals $1 \le i,j \le M$ and therefore $e_{max}(0) > 0$), then $\|e_\sigma(t)\| \to \infty$.*

Proof: Now, let $V_\sigma = \frac{1}{2}e_\sigma^\top e_\sigma$. Then, its derivative along the motion of the swarm is given by

$$\begin{aligned}\dot{V}_\sigma \;\; = \;\; & -\frac{A_\sigma}{Ml_\sigma}\sum_{i=1}^{M}\exp\left(-\frac{\|x^i-c_\sigma\|^2}{l_\sigma}\right)\|e_\sigma\|^2 \\ & -\frac{A_\sigma}{Ml_\sigma}\sum_{i=1}^{M}\exp\left(-\frac{\|x^i-c_\sigma\|^2}{l_\sigma}\right)e^{i\top}e_\sigma,\end{aligned}$$

where we used the fact that $x^i - c_\sigma = e^i + e_\sigma$. The case in which $A_\sigma > 0$ was proved in [1]. Here we will consider only the $A_\sigma < 0$ case.

From the above equation it can be shown that

$$\dot{V}_\sigma \;\; \ge \;\; \frac{|A_\sigma|}{Ml_\sigma}\sum_{i=1}^{M}\exp\left(-\frac{\|x^i-c_\sigma\|^2}{l_\sigma}\right)\|e_\sigma\|\Big[\|e_\sigma\| - e_{max}\Big],$$

which implies that if $\|e_\sigma\| > e_{max}$, we have $\dot{V}_\sigma > 0$. In other words, $\|e_\sigma\|$ will increase. From Lemma 3 we have that e_{max} is decreasing. Therefore, since by hypothesis $\|e_\sigma(0)\| > e_{max}(0)$ we have that $\dot{V}_\sigma > 0$ holds for all t. Now, given any large but fixed $D > 0$ and $\|e_\sigma(t)\| \le D$ we have

$$\begin{aligned}\exp\left(-\tfrac{\|x^i-c_\sigma\|^2}{l_\sigma}\right)\|e_\sigma\|\Big[\|e_\sigma\| - e_{max}\Big] \;\; & \ge \\ \exp\left(-\tfrac{(D^2+\varepsilon_3^2)}{l_\sigma}\right)D\Big[D-\varepsilon_3\Big] \;\; & > \;\; 0\end{aligned}$$

implying that

$$\dot{V}_\sigma \ge \frac{|A_\sigma|}{l_\sigma}\exp\left(-\frac{(D^2+\varepsilon_3^2)}{l_\sigma}\right)D\Big[D-\varepsilon_3\Big] > 0$$

and using a corollary of the Chetaev Theorem [7] we conclude that $\|e_\sigma\|$ will exit the D-neighborhood of c_σ. ■

Note that the result in Lemma 4 makes intuitive sense. If we have a valley (i.e., a minimum) it guarantees that the individuals will "gather" around it (as expected). If we have a hill (i.e., a maximum) and all the individuals are located on one side of the hill, it guarantees that the individuals diverge from it (as expected). If there is a hill, but the individuals are evenly spread around it, then we cannot conclude neither convergence nor divergence. This is because it can happen that the swarm may move to one side and diverge or the interindividual attraction forces can be counterbalanced by the interindividual repulsion combined with the repulsion from the hill so that the swarm does not move away from the hill.

Theorem 3 *Consider the swarm described by the model in Eq. (1) with interindividual attraction/repulsion function $g(\cdot)$ as given in Eq. (2). Assume that the σ-profile of the environment is given by Eq. (6). Then, as $t \to \infty$ we have*

- *If $A_\sigma > 0$, then all individuals $i = 1,\ldots,M$, will enter (and stay within) $B_{2\varepsilon_3}(c_\sigma)$, [1]*

- *If $A_\sigma < 0$ and $\|e_\sigma(0)\| \ge e_{max}(0)$, then all individuals $i = 1,\ldots,M$, will exit $B_D(c_\sigma)$ for any fixed $D > 0$.*

For the case $A_\sigma > 0$ Lemma 3 states that the swarm will have a maximum size of ε_3, i.e., $\|e_\sigma\| \le \varepsilon_3$ for all $i = 1,\ldots,M$, and Lemma 4 states that the swarm center will converge to the e_{max} and therefore to the ε_3 neighborhood of c_σ, i.e., $\|e_\sigma\| \le e_{max} \le \varepsilon_3$. Combining these two bounds we obtain the $2\varepsilon_3$ in the first case in Theorem 3.

Theorem 3 is a parallel of Theorem 2. However, here we have a weaker result since we cannot guarantee that $\bar{x} \to c_\sigma$ and we have a larger bound on the swarm size ($2\varepsilon_3$ here compared to ε_2 in Theorem 2).

6 Multimodal Gaussian Attractant/Repellent Profiles

Now, we will consider a profile which is a combination of Gaussian profiles. In other words, we consider the profile given by

$$\sigma(y) = -\sum_{i=1}^{N} \frac{A_\sigma^i}{2} \exp\left(-\frac{\|y - c_\sigma^i\|^2}{l_\sigma^i}\right) + b_\sigma, \qquad (7)$$

where $c_\sigma^i \in \mathbb{R}^n$, $l_\sigma^i \in \mathbb{R}^+$, $A_\sigma^i \in \mathbb{R}$ for all $i = 1, \ldots, N$, and $b_\sigma \in \mathbb{R}$. Note that since the A_σ^i's can be positive or negative there can be both hills and valleys leading to a "more irregular" profile. In [8], where social foraging was considered as an optimization process, a profile of this type was considered and convergence to minima of the profile was shown in simulation. For this profile we have the following results.

Lemma 5 *[1] Consider the swarm described by the model in Eq. (1) with interindividual attraction/repulsion function $g(\cdot)$ as given in Eq. (2). Assume that the σ-profile of the environment is given by Eq. (7). Then, as $t \to \infty$ for all individuals $i = 1, \ldots, M$, we have $x^i \to B_{\varepsilon_4}(\bar{x})$, for*

$$\varepsilon_4 = \frac{b(M-1)}{aM}\sqrt{\frac{c}{2}} \exp\left(-\frac{1}{2}\right) + \frac{1}{aM}\sum_{j=1}^{N} |A_\sigma^j| \sqrt{\frac{2}{l_\sigma^j}} \exp\left(-\frac{1}{2}\right).$$

Now, we have the following result, which is different from [1].

Lemma 6 *Consider the swarm described by the model in Eq. (1) with interindividual attraction/repulsion function $g(\cdot)$ as given in Eq. (2). Assume that the σ-profile of the environment is given by Eq. (7). Moreover, assume that for some $k, 1 \leq k \leq N$, we have*

$$\|x^i(0) - c_\sigma^k\| \leq h_k \sqrt{l_\sigma^k}$$

for some h_k and for all $i = 1, \ldots, M$, and that for all $j = 1, \ldots, N, j \neq k$ we have

$$\|x^i(0) - c_\sigma^j\| \geq h_j \sqrt{l_\sigma^j}$$

for some $h_j, j = 1, \ldots, N, j \neq k$ and for all $i = 1, \ldots, M$. (This means that the swarm is near c_σ^k and far from other $c_\sigma^j, j \neq k$.) Moreover, assume that

$$\frac{A_\sigma^k}{\sqrt{l_\sigma^k}} h_k \exp\left(-h_k^2\right) > \frac{1}{\alpha} \sum_{j=1, j\neq k}^{N} \frac{|A_\sigma^j|}{\sqrt{l_\sigma^j}} h_j \exp\left(-h_j^2\right),$$

is satisfied for some $0 < \alpha < 1$. Then, for $e_\sigma^k = \bar{x} - c_\sigma^k$ as $t \to \infty$ we will have

- *If $A_\sigma^k > 0$, then $\|e_\sigma^k(t)\| \leq \varepsilon_4 + \alpha h_k \sqrt{l_\sigma^k}$*
- *If $A_\sigma^k < 0$ and $\|e_\sigma^k(0)\| \geq e_{max}(0) + \alpha h_k \sqrt{l_\sigma^k}$, then $\|e_\sigma^k(t)\| \geq \varepsilon_4 + \alpha h_k \sqrt{l_\sigma^k}$, where $e_{max} = \max_{i=1,\ldots,M}\{e^i\}$.*

Proof: Let $V_\sigma^k = \frac{1}{2} e_\sigma^{k\top} e_\sigma^k$ be the Lyapunov function.

Case 1: $A_\sigma^k > 0$**:** Taking the derivative of V_σ^k along the motion of the swarm one can show that

$$\begin{aligned} \dot{V}_\sigma^k \leq & -\frac{A_\sigma^k}{M l_\sigma^k} \sum_{i=1}^{M} \exp\left(-\frac{\|x^i - c_\sigma^k\|^2}{l_\sigma^k}\right) \|e_\sigma^k\| \times \\ & \left[\|e_\sigma^k\| - e_{max} - \alpha h_k \sqrt{l_\sigma^k}\right], \end{aligned}$$

which implies that we have $\dot{V}_\sigma^k < 0$ as long as $\|e_\sigma^k\| > e_{max} + \alpha h_k \sqrt{l_\sigma^k}$, and from Lemma 5 we know that as $t \to \infty$ we have $e_{max}(t) \leq \varepsilon_4$.

Case 2: $A_\sigma^k < 0$**:** Similar to above, for this case it can be shown that

$$\begin{aligned} \dot{V}_\sigma^k \geq & \frac{|A_\sigma^k|}{M l_\sigma^k} \sum_{i=1}^{M} \exp\left(-\frac{\|x^i - c_\sigma^k\|^2}{l_\sigma^k}\right) \|e_\sigma^k\| \times \\ & \left[\|e_\sigma^k\| - e_{max} - \alpha h_k \sqrt{l_\sigma^k}\right], \end{aligned}$$

which implies that if $\|e_\sigma^k\| > e_{max} + \alpha h_k \sqrt{l_\sigma^k}$, we have $\dot{V}_\sigma > 0$. In other words, $\|e_\sigma^k\|$ will increase. From Lemma 5 we have that e_{max} is decreasing. Therefore, since by hypothesis $\|e_\sigma^k(0)\| > e_{max}(0) + \alpha h_k \sqrt{l_\sigma^k}$ we have that $\dot{V}_\sigma > 0$ holds at $t = 0$. Now, consider the boundary $\|e_\sigma^k\| = \varepsilon_4 + h_k \sqrt{l_\sigma^k}$. It can be shown that on the boundary we have

$$\dot{V}_\sigma \geq \frac{|A_\sigma^k| h_k (1-\alpha)\left(\varepsilon_4 + h_k \sqrt{l_\sigma^k}\right) \exp\left(-h_k^2\right)}{\sqrt{l_\sigma^k}} > 0,$$

from which once again using (a corollary to) the Chetaev Theorem we conclude that $\|e_\sigma^k\|$ will exit the $\varepsilon_4 + h_k \sqrt{l_\sigma^k}$-neighborhood of c_σ^k. ■

Now, using the results of the above two lemmas, i.e., Lemmas 5 and 6, we can state the following theorem.

Theorem 4 *Consider the swarm described by the model in Eq. (1) with interindividual attraction/repulsion function $g(\cdot)$ as given in Eq. (2) with linear attraction and bounded repulsion. Assume that the σ-profile of the environment is given by Eq. (7). Assume that the conditions of Lemma 6 hold. Then, as $t \to \infty$ all individuals will*

- *Enter the hyperball $B_{\varepsilon_5}(c_\sigma^k)$, where $\varepsilon_5 = 2\varepsilon_4 + \alpha h_k \sqrt{l_\sigma^k}$, if $A_\sigma^k > 0$, or*
- *Leave the $h_k \sqrt{l_\sigma^k}$-neighborhood of c_σ^k, if $A_\sigma^k < 0$.*

The only drawback of the above result is that we need

$$2\varepsilon_4 + \alpha h_k \sqrt{l_\sigma^k} < h_k \sqrt{l_\sigma^k}$$

in order for the result to make sense. This implies that we need

$$\varepsilon_4 < \left(\frac{1-\alpha}{2}\right) h_k \sqrt{l_\sigma^k}$$

which sometimes may not be easy to satisfy. Note, however, that for swarms with a large number of individuals (i.e., $M \to \infty$) we have $\varepsilon_4 \approx \varepsilon_1$ and this is easier to satisfy. One issue to note is that ε_4 (as well as the other bounds including ε_1 and ε_2) is a very conservative bound. In reality, the actual size of the swarm is typically much smaller than the bound. Therefore, effectively, ε_4 can be replaced with $e_{max}(\infty) < \varepsilon_4$ and the above condition may be satisfied more easily.

7 Analysis of Individual Behavior in a Cohesive Swarm

The results in the previous sections specify whether the swarm will diverge or converge, and if it converges they specify in which regions of the profile it will converge, together with bounds on the swarm size. However, they do not provide information about the ultimate behavior of the individuals. In other words, they do not specify whether the individuals will eventually stop moving or will end up in oscillatory motions within the specified regions. In this section, we will investigate this issue. In other words, we will analyze the ultimate behavior of the individuals in a quadratic profile with $A_\sigma > 0$, in a Gaussian profile with $A_\sigma > 0$, and in multimodal Gaussian profile for $A_\sigma^k > 0$ and with initial conditions and profile characteristics as in Lemma 6. Note that such analysis was not done in [1]. First, we define the state x of the system as the vector of the positions of the swarm members $x = [x^{1\top}, \ldots, x^{M\top}]^\top$. Let the invariant set of equilibrium points be

$$\Omega_e = \{x : \dot{x} = 0\}.$$

We will prove that for the above mentioned cases as $t \to \infty$ the state $x(t)$ converges to Ω_e, i.e., eventually all the individuals stop moving.

Theorem 5 *Consider the swarm described by the model in Eq. (1) with an attraction/repulsion function $g(\cdot)$ as given in Eq. (2). Assume that the σ-profile is one of the following*

- *A quadratic profile in Eq. (5) with $A_\sigma > 0$,*
- *A Gaussian profile in Eq. (6) with $A_\sigma > 0$, or*
- *A multimodal Gaussian profile in Eq. (7) with conditions of Lemma 6 for the $A_\sigma^k > 0$ case satisfied.*

Then, as $t \to \infty$ we have the state $x(t) \to \Omega_e$.

Proof: Choose the generalized Lyapunov function defined as

$$J(x) = \sum_{i=1}^{M} \sigma(x^i) + \frac{1}{2} \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} \left[a\|x^i - x^j\|^2 + bc\exp\left(-\frac{\|x^i - x^j\|^2}{c}\right)\right]$$

whose gradient at x^i is easily shown to be given by

$$\nabla_{x^i} J(x) = -\dot{x}^i.$$

Now, taking the time derivative of the Lyapunov function along the motion of the system we obtain

$$\dot{J}(x) = [\nabla_x J(x)]^\top \dot{x} = \sum_{i=1}^{M} [\nabla_{x^i} J(x)]^\top \dot{x}^i = -\sum_{i=1}^{M} \|\dot{x}^i\|^2 \leq 0,$$

for all t. Now, note that for all the cases in the hypothesis of the theorem, we have $J(x)$ bounded from below and the set defined as

$$\Omega_c = \{x : J(x) \leq J(x(0))\}$$

is compact and positively invariant with respect to the motions of the system. Then, we can apply LaSalle's Invariance Principle from which we conclude that as $t \to \infty$ the state x converges to the largest invariant subset of the set $\Omega \subset \Omega_c$ defined as

$$\Omega = \{x : \dot{J}(x) = 0\} = \{x : \dot{x} = 0\} = \Omega_e.$$

Since each point in Ω_e is an equilibrium, Ω_e is an invariant set and this proves the result. ∎

One issue to note here is that for the cases excluded in the above theorem, i.e., for the plane profile, quadratic profile with $A_\sigma < 0$, Gaussian profile with $A_\sigma < 0$, and the multimodal Gaussian profile for the $A_\sigma^k < 0$ case or $A_\sigma^k > 0$ case not necessarily satisfying the conditions of Lemma 6, the set Ω_c may not be compact. Therefore, we cannot apply LaSalle's Invariance Principle. Moreover, since they are diverging, intuitively we do not expect them to stop their motion. Furthermore, note that for the plane profile we have $\Omega_c = \emptyset$. In other words, there is no equilibrium for the swarm moving in a plane profile.

8 Simulation Examples

In this section we will provide some simulation examples to illustrate the theory in the preceding sections. We chose an $n = 2$ dimensional space for ease of visualization of the results and used the region $[0,30] \times [0,30]$ in the space. In all the simulations performed below we used $M = 11$ individuals. As parameters of the attraction/repulsion function $g(\cdot)$ in Eq. (2) we used $a = 0.01$, $b = 0.4$, and $c = 0.01$ in most of the simulations and $a = 0.1$ in one simulation. We performed simulations for all the profiles discussed in this article.

The upper left plot in Figure 2 is for a plane profile with $a_\sigma = [0.1, 0.2]^\top$. One easily can see that, as expected, individuals move along the gradient a_σ exiting the simulation region toward unboundedness. Note that initially some of the individuals move in a direction opposite to the negative gradient. This is because the interindividual attraction is much stronger than the intensity of the profile. In contrast, if we had a profile with intensity high enough to dominate the interindividual attraction, then we would not observe this type of motion. This, of course, does not mean that the swarm will not aggregate. As they move they will eventually aggregate as was shown in the preceding sections.

Next, consider a quadratic profile with extremum at $c_\sigma = [20,20]^\top$ and magnitude $A_\sigma = \pm 0.02$ (see the upper two plots in Figure 1). The upper left plot shows the paths of the individuals for the case $A_\sigma > 0$, whereas the plot on the upper right is for the $A_\sigma < 0$ case. Once more, we observe that the results support the analysis of preceding sections. Note also that the center $\bar{x}$ of the swarm converges to the minimum of the profile c_σ for the $A_\sigma > 0$ case and diverges from the maximum for the $A_\sigma < 0$ case (plots of which are not shown here for space limitations).

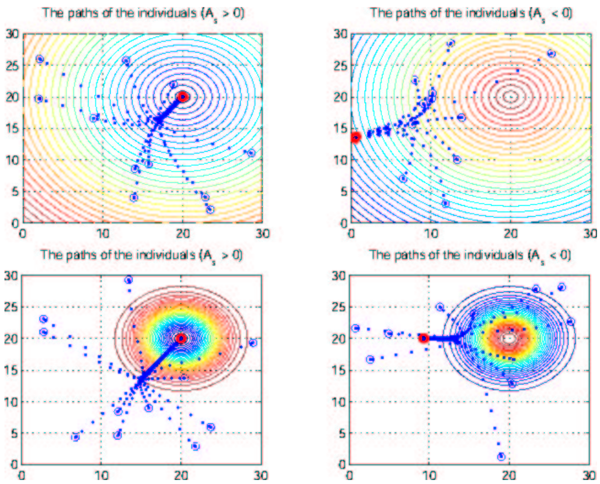

Figure 1: The response for a multimodal Gaussian profile.

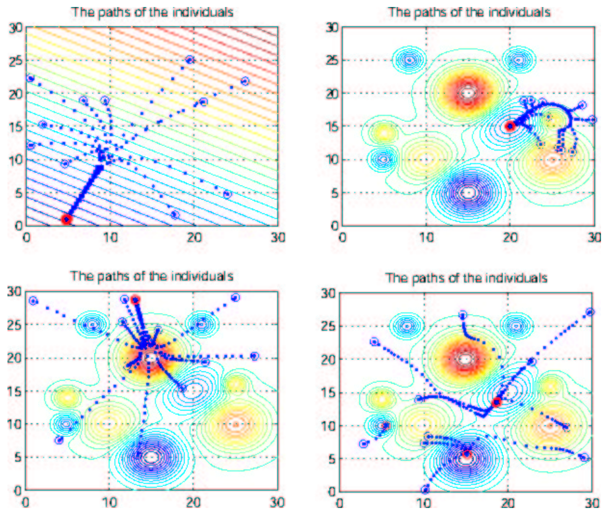

Figure 2: The response for a multimodal Gaussian profile.

Results of a similar nature were obtained also for the Gaussian profile as shown in the lower two plots in Figure 1. Once more we chose $c_\sigma = [20,20]^\top$ as the extremum of the profile. The other parameters of the profile were chosen to be $A_\sigma = \pm 2$ and $l_\sigma = 20$. Note that for the $A_\sigma > 0$ case, even though in theory we could not prove that $\bar{x} \to c_\sigma$, in simulations (not shown here) we observed that this is apparently the case. This was happening systematically in all the simulations that we performed.

In the simulation examples for the multimodal Gaussian profile we used a profile which has several minima and maxima, and the global minimum is located at $[15,5]^\top$ with a magnitude of 4 and a spread of 10. The upper right plot in Figure 2 shows an example run with initial member positions nearby a local minimum and shows convergence of the entire swarm to that minimum. The attraction parameter a was chosen to be $a = 0.01$ for this case. The lower left plot, on the other hand, illustrates the case in which we increased the attraction parameter to $a = 0.1$. You can see that the attraction is so strong that the individuals climb gradients to form a cohesive swarm (the extremum at $(15,10)$ is a maximum). For this and similar cases, the manner in which the overall swarm will behave (where it will move) depends on the initial position of the center $\bar{x}(0)$ of the swarm. For this run the center happened to be located on a region which caused the swarm to diverge. For some other simulation runs (not presented here) with different initial conditions the entire swarm converges to either a local or global minima. The lower left plot in Figure 2 shows a run for which we decreased the attraction parameter again to $a = 0.01$. For this case you can see that the swarm fails to form a cohesive cluster since the initial positions of the individuals are such that they move to a nearby local minima and the interindividual attraction is not strong enough to "pull them out" of these valleys. This causes formation of several groups or clusters of individuals at different locations of the space. For these reasons, the center $\bar{x}$ of the swarm does not converge to any minimum (as expected). Note, however, that Lemma 5 still holds, but ε_4 is large and includes all the region in which the individuals converge. Note also that during their motion to the groups, the individuals try to avoid climbing gradients and this results in motions resembling the motion of individuals in real biological swarms.

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