

## COORDINATED GRADIENT DESCENT: A CASE STUDY OF LAGRANGIAN DYNAMICS WITH PROJECTED GRADIENT INFORMATION

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**Abstract:** The paper studies gradient descent algorithms for vehicle networks. Each vehicle within the network is modeled as a double integrator in the plane. For each individual vehicle, the control input enabling coordinated gradient descent consists of a gradient descent control term and additional inter-vehicle forcing terms. When each vehicle has enough sensors to measure the full gradient at its current position, then the closed-loop system becomes Lagrangian. We focus in the present paper upon the more practical situation where each vehicle has only one sensor with which to sample the environment. We take this into account by replacing the full gradient in the closed-loop equations by its projection on the direction of motion for each individual vehicle. This gives rise to a differential equation with *discontinuous* right-hand side. In order to avoid the (practical and theoretical) complications that arise as a consequence of these discontinuities, we modify the inter-vehicle forcing terms and represent the velocity of each vehicle by a magnitude and an angle, resulting in a set of *smooth* differential equations. We demonstrate our approach with simulations. Copyright © 2003 IFAC

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## 1. INTRODUCTION

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We propose a coordinated control strategy for multi-vehicle gradient descent (or ascent) in a sampled environment. Vehicle networks that can efficiently climb gradients are of great interest in missions such as search and map where a spatially distributed environmental signal is to be mapped or its source is to be found. Such vehicle networks could be used, for example, to locate hydrothermal vents deep in the sea by climbing the associated mineral plume and/or temperature gradient.

A vehicle network has a number of important advantages over a single large vehicle. The large vehicle could be outfitted with distributed sensors so that local gradients could be computed. However, the sensor array would then be rigid and therefore there would be little ability to adapt the array configuration to the environment. Furthermore, failure of the vehicle would mean failure of the entire mission. A group of smaller vehicles, on the other hand, could provide a reconfigurable distributed sensor array. A successful coordinated control strategy would enable the vehicles to perform as a network that could change shape and maneuver in response to the measured environment.

Gradient following by autonomous vehicle systems inspired by bacterial chemotaxis has been explored by Burian *et al.* (1996) and Hoskins (1995). Burian *et al.* (1996) use the Autonomous Benthic Explorer (ABE) to find the deepest spot in a lake following the *run and tumble* behavior of flagellated bacteria like the *Escherichia coli* as described by Adler (1966) and Berg (1983). Hoskins (1995) applies the chemotaxis approach together with some basic agent interactions to a multiple agent problem. Gazi and Passino (2002) take an approach similar to ours in which individuals balance their own gradient descent with inter-vehicle attraction and repulsion terms. The approach of Gazi and Passino (2002) differs from ours, however, in that individuals are modeled with kinematic equations (i.e., velocity inputs rather than forces), it is assumed that the gradient of the environmental field is known at the individual's position, and each individual must know the position of every member in the group.

Bachmayer and Leonard (2001; 2002) first outlined the gradient climbing approach where each vehicle within the network uses control forces that consist of an approximation of the local gradient and additional inter-vehicle control forces derived from artificial potentials. The approximation of the local gradient is based on a single sensor per vehicle: each vehicle is assumed to be able to measure the gradient only in the direction of motion. The inter-vehicle forces not only contribute to maintaining a uniformly spaced vehicle network, but also provide the necessary implicit communication to drive the group as a whole to the global minimum (or maximum) of the sampled environmental field.

In Section 2 we discuss the coordinated strategy for gradient descent in the case that each vehicle can measure the full gradient at its current position. In Section 3 we consider the case that each vehicle can only measure the gradient in its direction of motion leading to a coordinated gradient descent strategy with projected gradient

information. In Sections 4 and 5 we introduce a modification of the projected gradient descent equations of the previous section and study some of its properties. We demonstrate our approach with simulations.

## 2. GRADIENT DESCENT

Our goal is to enable a network of  $N$  vehicles to locate minima of an environmental variable by coordinated gradient descent. For the purpose of the present study, we restrict attention to gradient descent in a planar environment and model each vehicle as a point mass with fully actuated dynamics

$$\ddot{x}_i = u_i, \quad x_i, u_i \in \mathbb{R}^2. \quad (1)$$

where the subscript  $i$  refers to agent  $i$ . The environmental variable is assumed to be time-invariant and is modeled by a smooth function  $T : \mathbb{R}^2 \rightarrow \mathbb{R}$ .

Bachmayer and Leonard (2002, Section 3) propose a feedback controller

$$u_i = -k_d \dot{x}_i - k \nabla T(x_i) + \sum_{j=1, j \neq i}^N F_{ij}, \quad (2)$$

with  $k$  and  $k_d$  positive constants. In this feedback controller,  $-k \nabla T(x_i)$  is a gradient descent control term and  $F_{ij}$  is a force acting on vehicle  $i$  generated by vehicle  $j$ . These inter-vehicle forces  $F_{ij}$  are included to enforce *coordinated* gradient descent, and are derived from a scalar, inter-vehicle artificial potential (Leonard and Fiorelli, 2001)  $V : \mathbb{R}^2 \rightarrow \mathbb{R}$  according to

$$F_{ij} = -\nabla V(x_i - x_j).$$

We assume that the inter-vehicle potential  $V$  is rotationally symmetric, which implies that the force on vehicle  $i$  generated by vehicle  $j$  equals minus the force on vehicle  $j$  generated by vehicle  $i$ :

$$\nabla V(x_i - x_j) + \nabla V(x_j - x_i) = 0.$$

With this controller, the closed-loop system becomes a (damped) Lagrangian system with kinetic energy  $\sum_{i=1}^N \|\dot{x}_i\|^2/2$  and potential energy  $k \sum_{i=1}^N T(x_i) + \sum_{i=1}^{N-1} \sum_{j=i+1}^N V(x_i - x_j)$ . Taking the sum of kinetic and potential energy as a candidate Lyapunov function  $\mathcal{V}$  and evaluating its time-derivative along the solutions of the closed-loop system, we obtain the inequality

$$\dot{\mathcal{V}} = -k_d \sum_{i=1}^N \|\dot{x}_i\|^2 \leq 0. \quad (3)$$

If the potential energy is radially unbounded as a function of  $(x_1, \dots, x_N)$ , then an application of LaSalle's invariance principle yields convergence of the vehicle network to the set of equilibria

where  $\dot{x}_i = 0$  and  $k\nabla T(x_i) + \sum_{j=1, j \neq i}^N \nabla V(x_i - x_j) = 0$  for each  $i$ .

Based upon these Lyapunov/LaSalle arguments, Bachmayer and Leonard (2002) show by means of various examples that the vehicle network is indeed able to locate local minima of the environmental variable  $T$  through coordinated gradient-descent.

The above controller addresses gradient descent of vehicle networks in a full gradient information context, assuming that each vehicle is equipped with enough sensors to measure the gradient at its current position. In the present paper we are interested in the more practical situation of vehicles that each have only one sensor with which to sample the environment. In this case, each vehicle has only partial information about the local gradient at its current position. By sampling the environmental field  $T$  along its path, vehicle  $i$  can estimate the directional derivative of  $T$  in the direction of  $\dot{x}_i$ . We may thus assume that the directional derivative  $\nabla T(x_i) \cdot \dot{x}_i$  is available for feedback.

In this paper we present several modified gradient descent algorithms which rely only upon information about the directional derivative. Anticipating the results to follow, we mention that these modified gradient descent schemes give rise to the same Lyapunov balance as in the full gradient case; thus suggesting that these modified vehicle networks indeed may be able to successfully perform gradient descent.

### 3. GRADIENT DESCENT WITH PROJECTED GRADIENT INFORMATION

The first and probably most natural modification of the original gradient descent algorithm has been outlined by Bachmayer and Leonard (2002) and is obtained by replacing the full gradient  $\nabla T(x_i)$  in (2) by the projected gradient  $\nabla_p T(x_i)$ , which is defined for all  $\dot{x}_i \neq 0$  as follows:

$$\nabla_p T(x_i) = \left( \nabla T(x_i) \cdot \frac{\dot{x}_i}{\|\dot{x}_i\|} \right) \frac{\dot{x}_i}{\|\dot{x}_i\|}. \quad (4)$$

The projected gradient is the projection of the full gradient  $\nabla T(x_i)$  on the direction of motion (Fig. 1). The modified closed-loop dynamics are

$$\ddot{x}_i = -k_d \dot{x}_i - k \nabla_p T(x_i) - \sum_{j=1, j \neq i}^N \nabla V(x_i - x_j). \quad (5)$$

Of course, this differential equation only makes sense as long as all  $\dot{x}_i$  are different from zero, as the projected gradient has only been defined for nonzero  $\dot{x}_i$ . There is, however, no guarantee that the velocities of the vehicles will be nonzero for all

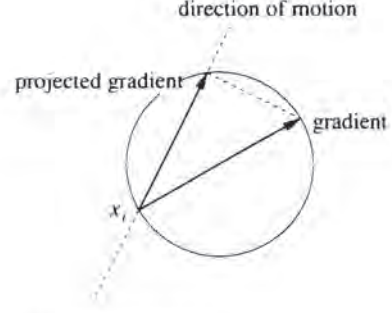


Fig. 1. Projected gradient

times, even if all initial velocities are nonzero. In order to give a meaning to the differential equation (5) for zero  $\dot{x}_i$ , we introduce the differential inclusion which arises by giving a set-valued interpretation to the projected gradient when  $\dot{x}_i = 0$ . We define, for  $\dot{x}_i = 0$ ,

$$\nabla_p T(x_i) = \{(\nabla T(x_i) \cdot e) e : e \in \mathbb{R}^2 \text{ with } \|e\| = 1\}. \quad (6)$$

In this case, the projected gradient corresponds to all those vectors whose end points are located on the circle featuring in Fig. 1. With this definition, the differential equation (5) may be interpreted as the differential inclusion (with an abuse of notation)

$$\ddot{x}_i \in -k_d \dot{x}_i - k \text{co}(\nabla_p T(x_i)) - \sum_{j=1, j \neq i}^N \nabla V(x_i - x_j), \quad (7)$$

where  $\text{co}(\nabla_p T(x_i))$  denotes the smallest convex set containing  $\nabla_p T(x_i)$ . The right-hand side of this differential inclusion is a nonempty-, compact- and convex-valued set-valued function which is upper semi-continuous. It is known that, because of these properties, the differential inclusion (7) has solutions for all possible initial conditions (Filippov, 1988, Theorems 1 and 2, pp. 77-78).

*Theorem 1.* The time-derivative of the Lyapunov function  $\mathcal{V}$  along the solutions of the differential inclusion (7) is a well-defined function of  $(x_1, \dots, x_N, \dot{x}_1, \dots, \dot{x}_N)$  and satisfies

$$\dot{\mathcal{V}} = -k_d \sum_{i=1}^N \|\dot{x}_i\|^2 \leq 0.$$

These considerations suggest that the proposed dynamics may give rise to cooperative gradient descent for the vehicle network. There are, however, a couple of problems with the proposed dynamics. First of all, the physical interpretation of the solutions of the differential inclusion (7) is problematic since, in general, this differential inclusion does not have unique solutions. Furthermore, the Lyapunov balance suggests that the vehicle network will eventually converge to the set of discontinuity where all  $\dot{x}_i$  are zero. This is reflected by the

fact that a numerical simulation of the differential equation (5) is problematic and it suggests that we may face practical problems when trying to implement this gradient descent scheme in practice. Finally, a theoretical (stability) analysis of the dynamics around the equilibrium configurations is complicated by the discontinuities which are inherent to the present gradient descent scheme. In order to deal with all of these problems, we propose in the present paper a second, different modification of the original gradient descent algorithm, which is applicable in the context of sampling the environment with one sensor for each vehicle. In addition to replacing the full gradient by the projected gradient, we also introduce a new representation for the velocities and we modify the inter-vehicle forcing terms. This is the subject of the following section.

#### 4. LAGRANGIAN DYNAMICS WITH PROJECTED GRADIENT INFORMATION

When each vehicle samples the environmental field  $T$  along its path, it is convenient to represent the velocity  $\dot{x}_i \in \mathbb{R}^2$  of the  $i$ -th vehicle by an angle variable  $\phi_i \in S^1$  (the direction of the velocity) and a real number  $v_i \in \mathbb{R}$  (the magnitude of the velocity), related to  $\dot{x}_i \in \mathbb{R}^2$  by

$$\dot{x}_i = (v_i \cos(\phi_i), v_i \sin(\phi_i)). \quad (8)$$

We refer to the couple  $(v_i, \phi_i)$  as *quasi-polar coordinates* for the velocity  $\dot{x}_i$ . The prefix ‘quasi’ refers to the fact that the magnitude  $v_i$  is allowed to be any real number, not necessarily positive. As a consequence, the mapping from  $(v_i, \phi_i)$  to  $\dot{x}_i$  is not invertible and we have to be very careful with the interpretation of this representation. We postpone these interpretation issues until the end of this section.

In the present case of sampling  $T$  along the paths of the vehicles, we replace the full gradient equations (1)–(2) by the following equations, which depend on the derivative of  $T$  in the direction of the velocity only:

$$\dot{x}_i = (v_i \cos(\phi_i), v_i \sin(\phi_i)). \quad (9)$$

$$\begin{aligned} \dot{v}_i = & -k_d \dot{v}_i - k \nabla T(x_i) \cdot (\cos(\phi_i), \sin(\phi_i)) \\ & - \sum_{j=1, j \neq i}^N \nabla V(x_i - x_j) \cdot (\cos(\phi_i), \sin(\phi_i)), \end{aligned} \quad (10)$$

$$\dot{\phi}_i = - \sum_{j=1, j \neq i}^N \nabla V(x_i - x_j) \cdot (-\sin(\phi_i), \cos(\phi_i)). \quad (11)$$

Equations (9)–(11) form a smooth dynamical system on the manifold  $(\mathbb{R}^3 \times S^1)^N$ . The smoothness

of the right-hand sides of (9)–(11) should be contrasted with the discontinuities that were inherent to the projected gradient equations of the previous section. The importance of the current smoothness property should not be underestimated. It guarantees uniqueness of solutions, which is crucial for a meaningful physical interpretation, and it makes the set of equations amenable for standard analysis tools, such as the linearization principle.

Using the relation (8), we may project equations (9)–(11) onto the cartesian space  $\mathbb{R}^{4N}$  resulting in the differential inclusion (with an abuse of notation)

$$\begin{aligned} \ddot{x}_i \in & -k_d \dot{x}_i - k \nabla_p T(x_i) - \sum_{j=1, j \neq i}^N \nabla_p V(x_i - x_j) \\ & \pm |\dot{x}_i| \sum_{j=1, j \neq i}^N \nabla_\perp V(x_i - x_j), \end{aligned} \quad (12)$$

where  $\nabla_p V(\cdot)$  and  $\nabla_\perp V(\cdot)$  are defined as follows:

$$\begin{aligned} \nabla_p V(\cdot) &= \left( \nabla V(\cdot) \cdot \frac{\dot{x}_i}{\|\dot{x}_i\|} \right) \frac{\dot{x}_i}{\|\dot{x}_i\|}, \\ \nabla_\perp V(\cdot) &= \nabla V(\cdot) - \nabla_p V(\cdot), \end{aligned}$$

when  $\dot{x}_i \neq 0$ , and as follows:

$$\begin{aligned} \nabla_p V(\cdot) &= \nabla_\perp V(\cdot) \\ &= \{(\nabla V(\cdot) \cdot e) e : e \in \mathbb{R}^2 \text{ with } \|e\| = 1\} \end{aligned} \quad (13)$$

when  $\dot{x}_i = 0$ . In words,  $\nabla_p V(\cdot)$  and  $\nabla_\perp V(\cdot)$  are, respectively, the projection of  $\nabla V(\cdot)$  on the direction of motion and on the direction perpendicular to the direction of motion. The right-hand side of the differential inclusion (12) is constructed from the right-hand side of the differential equations (9)–(11) via the projection (8) through the process of taking the pointwise push-forward of individual vectors. The plus-minus sign in the right-hand side of (12) arises from the fact that each nonzero velocity vector  $\dot{x}_i$  has two representations in quasi-polar coordinates, one with positive magnitude and one with negative magnitude. The set-valued nature of the projected gradients when  $\dot{x}_i = 0$  arises from the fact that  $\dot{x}_i = 0$  has infinitely many representations in quasi-polar coordinates, corresponding to  $v_i = 0$  and arbitrary  $\phi_i \in S^1$ . The following result is an immediate consequence of the preceding construction.

*Theorem 2.* Every solution of the differential equations (9)–(11) projects, via the relation (8), to a solution of the differential inclusion (12).

Comparing the differential inclusion (12) with the gradient descent equation (7) from the previous section, we clearly see that the inter-vehicle forcing terms have been modified for the present gradient descent scheme. The components of the

inter-vehicle forcing terms which are orthogonal to the velocities of the vehicles have been multiplied by a factor  $\pm|\dot{x}_i|$ .

When we introduced quasi-polar coordinates for the velocities of the agents in the beginning of this section, we mentioned that we have to be very careful with the interpretation of this coordinate representation. We now discuss these interpretation issues in some depth. A problem may arise when the dynamics (9)–(11) give rise to a trajectory along which, during some time-interval and for some vehicle  $i$ ,  $v_i$  is identically zero and  $\phi_i$  changes value. This leads to interpretation problems, since the quasi-polar dynamics assume knowledge of  $\nabla T(x_i) \cdot (\cos(\phi_i), \sin(\phi_i))$ , which is physically not available when  $\phi_i$  is changing while the vehicle is not moving. Fortunately, it is to be expected that this is a very unlikely situation. Indeed, it is to be expected that, at least generically, the set of initial conditions that give rise to such trajectories is of measure zero. This follows from the fact that having a trajectory along which  $v_i$  and  $\dot{v}_i$  vanish simultaneously at some time instant for some vehicle  $i$  is already expected to be a degenerate situation. Indeed, based upon dimension counting arguments, it is to be expected that the set  $\{v_i = \dot{v}_i = 0\}$  has dimension  $4N - 2$  and thus that the set of initial conditions from which the above set can be reached has dimension  $4N - 1$ , and thus has measure zero. Of course, the above arguments are only intuitive. A rigorous treatment of these interpretation issues is a subject for further research.

## 5. STABILITY PROPERTIES OF THE LAGRANGIAN DYNAMICS WITH PROJECTED GRADIENT INFORMATION

We are interested in the extent to which the quasi-polar gradient descent scheme indeed enables a vehicle network to locate minima of an environmental field via coordinated gradient descent.

A first observation in this regard concerns the Lyapunov function

$$\tilde{V} = \sum_{i=1}^N v_i^2/2 + k \sum_{i=1}^N T(x_i) + \sum_{i=1}^{N-1} \sum_{j=i+1}^N V(x_i - x_j), \quad (14)$$

which represents the sum of kinetic and potential energy in quasi-polar velocity coordinates, and corresponds to the original Lyapunov function  $V$  on  $\mathbb{R}^{4N}$ .

*Theorem 3.* The time-derivative of the Lyapunov function  $\tilde{V}$  along the solutions of the differential equations (9)–(11) satisfies

$$\dot{\tilde{V}} = -k_d \sum_{i=1}^N v_i^2 \leq 0. \quad (15)$$

What does this Lyapunov balance tell us about the asymptotic properties of the vehicle network? Recall that the equivalent Lyapunov balance (3) on  $\mathbb{R}^{4N}$  played an instrumental role in proving that vehicle networks are able to locate minima of  $T$  in a full-gradient information context (Bachmayer and Leonard, 2002). Theorem 3 thus suggests that the proposed quasi-polar dynamics may indeed give rise to cooperative gradient descent enabling the vehicle network to locate minima of the environmental field  $T$ .

However, this need not necessarily be true. Indeed, the time-derivative of the Lyapunov functions  $V$  and  $\tilde{V}$  are only negative *semi*-definite. Hence, LaSalle's principle needs to be invoked in addition to those Lyapunov arguments. LaSalle's principle enables us to conclude convergence to the largest invariant set contained in  $\{\dot{V} = 0\}$  respectively  $\{\dot{\tilde{V}} = 0\}$ . In the present context, where  $\dot{V}$  and  $\dot{\tilde{V}}$  are given by (3) and (15) respectively, LaSalle's principle enables us to conclude convergence to the set of equilibrium points. However, the quasi-polar dynamics (9)–(11) give rise to many more equilibria than the original gradient descent algorithm (1)–(2), and there is, in general, no guarantee that the combined basin of attraction of all these additional equilibrium points has zero measure. Accordingly, the convergence properties of the quasi-polar gradient scheme in a sampled environment may differ significantly from the convergence properties of the original gradient descent scheme in a full gradient context.

This is very well illustrated by the single vehicle case  $N = 1$ . Let us, for example, consider a single vehicle in a quadratic environmental field given by  $T(x) = \|x\|^2$ . Clearly, in a full gradient information context, the single vehicle will converge exponentially to the unique minimum at the origin. In a sampled environment, however, when implementing the quasi-polar gradient descent scheme (9)–(11), a single vehicle will, in general, not converge to the minimum. Indeed, since there is no inter-vehicle force acting on the vehicle, the vehicle only feels a projected gradient descent control term acting along the direction of motion, and thus the vehicle is constrained to move on a straight line determined by its initial velocity direction. The quasi-polar dynamics will enable the single vehicle to converge to the relative minimum of  $T$  along this constraint line (Fig. 2).

We thus see that, with the quasi-polar gradient descent algorithm, a single vehicle is, in general, not able to locate the minima of an environmental field  $T$ . This may be interpreted as follows: by

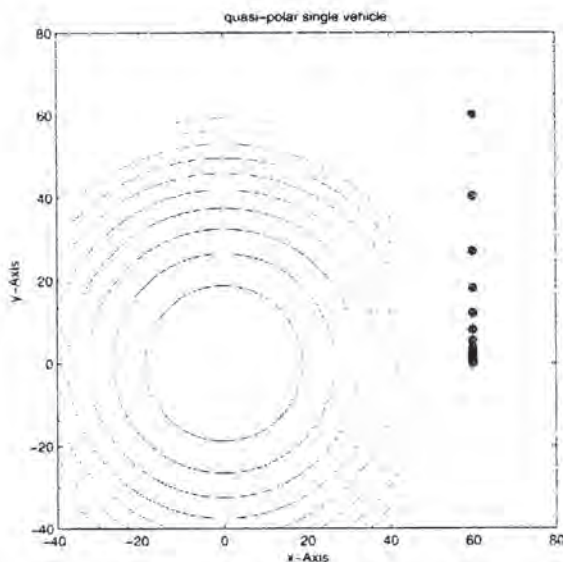


Fig. 2. Gradient descent by a single vehicle sampling the environment. Simulation of equations (9)–(11) with  $T(x) = \|x\|^2$ ,  $k = 0.2$ ,  $k_d = 10$ . Since there is only one vehicle, there is no need to specify the inter-vehicle potential  $V$ . The initial velocity of the vehicle is given by  $(v, \phi) = (1, \pi/2)$ .

sampling the environmental field along its path, a single vehicle does not gather enough information to determine the full gradient at its current position.

The above situation of a single vehicle should be contrasted with the multiple vehicle case. Unlike a single vehicle, a network of multiple vehicles may be able to locate minima of an environmental field via the quasi-polar gradient descent algorithm. This is illustrated in Fig. 3. It is important to stress that the exchange of information between individual vehicles, which enables the vehicle network to perform gradient descent, is not achieved through explicit communication. Instead, the necessary information exchange is a consequence of the implicit communication that arises from the inter-vehicle forces keeping the vehicles in formation.

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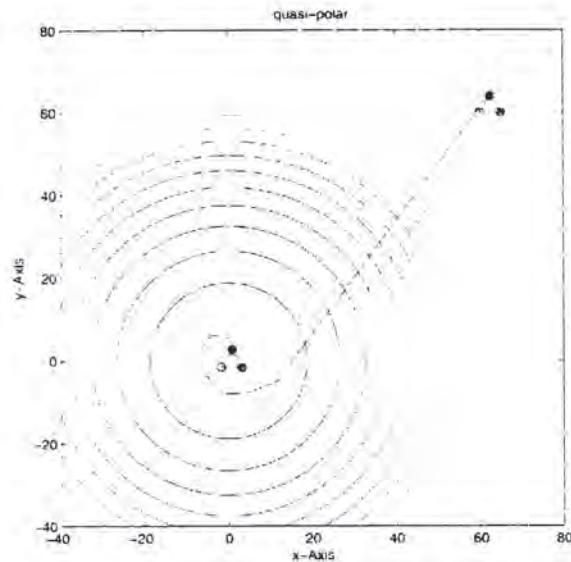


Fig. 3. Gradient descent by a network of three vehicles sampling the environment. Simulation of equations (9)–(11) with  $T(x) = \|x\|^2$ ,  $k = 0.2$ ,  $k_d = 10$ ,  $V(x) = 50(\|x\| - 5)^2$ . The initial velocities of the vehicles are given by  $(v_i, \phi_i) = (1, \pi/2)$  for all  $i = 1, 2, 3$ .

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