SOLUTIONS TO FIRST EXAM-C

1. $\begin{bmatrix} 1 & -2 & 2 & -6 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 2 & -4 & 1 & 3 & 0 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad \begin{array}{rcrcrcl} x_1 & - & 2x_2 & & & + & 4x_4 = 0 \\ & & & & x_3 & - & 5x_4 = 0, \\ & & & & & & 0 = 0 \end{array}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 - 4x_4 \\ x_2 \\ 5x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 5 \\ 1 \end{bmatrix}, \qquad x_2 \text{ and } x_4 \text{ free variables.}$$

2. a. Span$\{\mathbf{u}, \mathbf{v}\}$ is a plane in $\mathbb{R}^3$ through $\mathbf{u}, \mathbf{v}$ and the origin.

b. We want to know if the equation $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{w}$ has a solution. Row reduce the augmented matrix for this equation:

$$\begin{bmatrix} 1 & 3 & -2 \\ -1 & 1 & 2 \\ -2 & -6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \quad \text{The equation is } 0x_1 + 0x_2 = 2 \text{ is impossible.}$$

So the vector equation $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{w}$ has no solution, and $\mathbf{w}$ is not in Span$\{\mathbf{u}, \mathbf{v}\}$.

c. The set Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is larger than Span$\{\mathbf{u}, \mathbf{v}\}$ because it must contain $\mathbf{w}$ and $\mathbf{w}$ is not in Span$\{\mathbf{u}, \mathbf{v}\}$. In fact, if $A = [\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}]$, then the row reduction above shows that A has three pivot positions, one in each row. By a theorem, the columns of A span $\mathbb{R}^3$.

3. a. By inspection, neither column of A is a multiple of the other, so the columns are linearly independent.

b. One column of A is the zero vector. The columns are linearly dependent, by a theorem that says a set is linearly dependent if it contains the zero vector.

5. If $T(\mathbf{x}) = A\mathbf{x}$ and $T\colon \mathbb{R}^5 \mapsto \mathbb{R}^7$, then A must have 7 rows and 5 columns.

4. a. ... is a linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights.

b. ... the equation $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = 0$ has only the trivial solution.

c. (i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$, and
(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in $\mathbb{R}^n$ and all scalars c.

6. A 4×3 matrix can have at most 3 pivot positions, because it has only 3 columns. Thus A cannot have a pivot position in each of its 4 rows. By a theorem, the columns of A cannot span $\mathbb{R}^4$.

7. a. Daily output vectors: Order vector:

(mine #1) $\mathbf{v}_1 = \begin{bmatrix} 20 \\ 550 \end{bmatrix} \begin{matrix} \text{(metric tons of copper)} \\ \text{(kilograms of silver)} \end{matrix}$, (mine #2) $\mathbf{v}_2 = \begin{bmatrix} 30 \\ 500 \end{bmatrix}$, $\quad \mathbf{b} = \begin{bmatrix} 150 \\ 2825 \end{bmatrix}$.

Let x_1 be days of operation of mine #1 and x_2 days of operation of mine #2. Then the problem is to find x_1 and x_2 to satisfy the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{b} \quad \left(\text{or:} \quad x_1 \begin{bmatrix} 20 \\ 550 \end{bmatrix} + x_2 \begin{bmatrix} 30 \\ 500 \end{bmatrix} = \begin{bmatrix} 150 \\ 2825 \end{bmatrix}\right)$$

b. The equivalent matrix equation is (there are several possible answers):

$$[\mathbf{v}_1 \quad \mathbf{v}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{b}, \quad \text{or} \quad \begin{bmatrix} 20 & 30 \\ 550 & 500 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 150 \\ 2825 \end{bmatrix}$$

$$\text{or} \quad A\mathbf{x} = \mathbf{b}, \quad \text{where} \quad A = \begin{bmatrix} 20 & 30 \\ 550 & 500 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note: A common incorrect answer is the augmented matrix $\begin{bmatrix} 20 & 30 & 150 \\ 550 & 500 & 2825 \end{bmatrix}$, which represents a system of two linear equations, but is not what we have called a "matrix equation". See Theorem 3, page 41.